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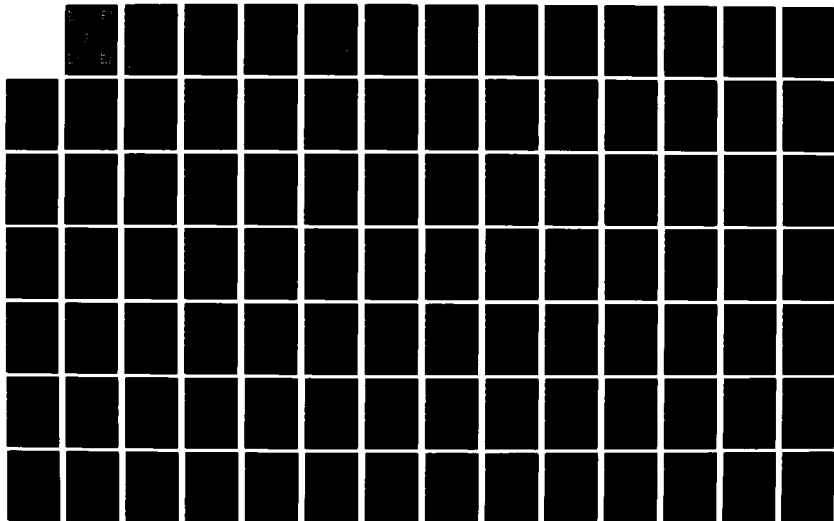
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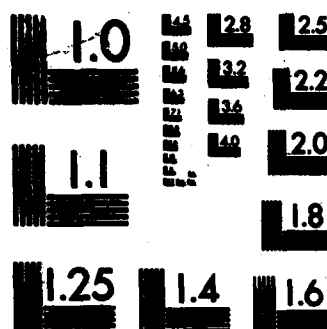
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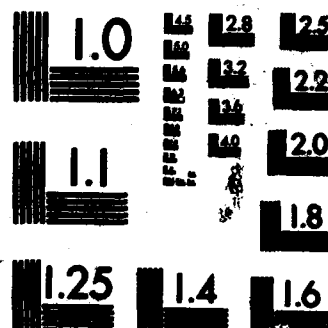
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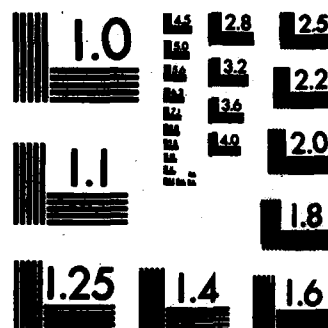




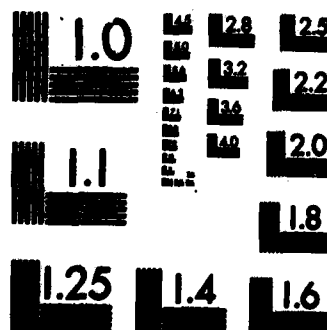
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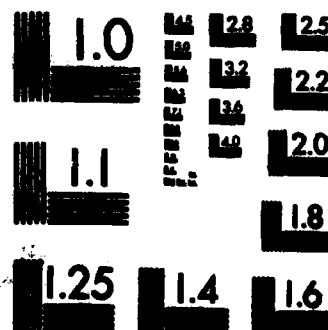
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CURRENT METHODOLOGIES FOR THE ANALYSIS OF CONTINGENCY

TABLES: ROBUSTNESS WITH RESPECT TO SMALL EXPECTED VALUES

Major Rickey Arthur Kolb

Final Report 9 June 1982

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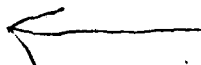
**A thesis submitted to Georgia Institute of Technology, Atlanta, Georgia
in partial fulfillment of the requirements for the degree of Doctor of
Philosophy in the School of Industrial and Systems Engineering**

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO. A120805	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Current Methodologies for the Analysis of Contingency Tables: Robustness with Respect to Small Expected Values		5. TYPE OF REPORT & PERIOD COVERED Final 9 June 1982
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Major Rickey Arthur Kolb		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Student, HQDA, MILPERCEN (DAPC-OPP-E) 200 Stovall Street Alexandria, VA 22332		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS HQDA, MILPERCEN, ATTN: DAPC-OPP-E, 200 Stovall Street Alexandria, VA 22332		12. REPORT DATE 9 June 1982
		13. NUMBER OF PAGES 342
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNC
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES Thesis Georgia Institute of Technology		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Contingency Tables, Pearson Chi-Square, Kullback Minimum Discrimination Information, Grizzle, Starmer and Koch (GSK) Statistic, Small Expected Values, Multinomial Sampling, Log-Linear Model, Minimum Cell Expectation, Critical Expected Value Distributions.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The primary purpose of this study is to investigate the robustness characteristics of some of the current methodologies for the analysis of contingency tables as the size of the table increases. A Monte Carlo simulation provides estimates of exact significance levels for comparison with nominal levels. An extensive array of tables, probability designs, and sample sizes are used.		

Three asymptotic chi-squared statistics are evaluated: the Pearson statistic, the Kullback minimum discrimination information statistic, and the Grizzle, Starmer, and Koch (GSK) Wald statistic. A 95% confidence interval criterion about the nominal levels is defined, and certain minimum sample sizes (N_m) are determined. These N_m provide the parameters for calculation of critical expected value (CEV) distributions. The CEV distributions are extensively analyzed.

The GSK statistic is shown to be highly conservative, significantly biased by the presence of zero cells, and less robust as the size of the table increases. The Kullback statistic is shown to be generally liberal with best performance at the lower nominal levels, and little change in robustness characteristics is noted. The Pearson statistic is shown to be somewhat conservative and clearly superior to the other statistics, and the robustness with respect to small expected values is significantly improved as the size of the table increases. 

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CURRENT METHODOLOGIES FOR THE ANALYSIS OF CONTINGENCY

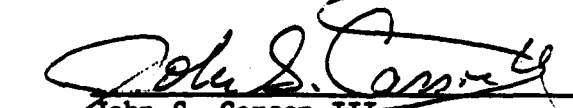
TABLES: ROBUSTNESS WITH RESPECT TO SMALL EXPECTED
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Date approved by Chairman 7/9/82

ACKNOWLEDGMENTS

No one can complete a Ph.D. without the assistance, guidance, and encouragement of many people. I wish to thank Dr. William W. Hines who first encouraged me to pursue a Ph.D. degree. Likewise, I am grateful to Col. Jack M. Pollin, Head of the Department of Mathematics, USMA, for his encouragement and allowing me to return to Georgia Tech to begin this research.

I wish to thank my principal advisor Dr. Douglas C. Montgomery for his continuing guidance and assistance throughout this research and Dr. James W. Walker who made significant contributions and listened patiently to my ideas at each step of the research. Also, I wish to thank Dr. Harry M. Wadsworth for his timely suggestions and the other members of my reading committee, Dr. Russell G. Heikes and Dr. John S. Carson III.

No Ph.D. candidate could pursue his research without the assistance of the library staff. I wish to give a special thanks to Mrs. Celeste B. Sproul as representative of the many library personnel who have helped me at Georgia Tech, West Point, and Ft. Leavenworth.

Lastly, I wish to thank my wife, Mary, whose constant encouragement, support, and understanding made it possible for me to devote those extra concentrated hours needed to complete this research. Besides that, she explained to our four sons, Richard, Patrick, Robert, and Edward, why their Daddy was still in school.

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SUMMARY

Current methodologies for the analysis of contingency tables are being used on larger and larger tables with little concern for the classic problem of small expected values. The inference is that for these larger tables these methodologies are more robust with respect to small expected values. Previous studies of this small expected value problem have been limited as to the statistics investigated, the hypotheses considered, and the assessment criteria and table designs used. There has been little evidence of robustness trends as the table size increases.

The primary purpose of this study is to investigate the robustness characteristics of some of the current methodologies for the analysis of contingency tables as the size of the table increases. A Monte Carlo simulation provides estimates of exact significance levels for comparison with nominal levels. An extensive array of tables, probability designs, and sample sizes are used. The underlying sampling model is multinomial. The log-linear model is used to define the variable relationships and provide independence hypothesis test parameters. Three asymptotic chi-squared statistics are evaluated: the Pearson statistic, the Kullback minimum discrimination information statistic, and the Grizzle, Starmer, and Koch (GSK) Wald statistic.

A minimum cell expectation (MCE) criterion is used to show that the performance of the statistics generally improves as the MCE in-

creases. The second moment of the probability structure is shown to be a useful parameter for measuring the performance of each statistic. A 95% confidence interval criterion about the nominal levels is defined, and certain minimum sample sizes (N_m) are determined for each table, statistic, probability design, and nominal level combination. These N_m provide the parameters for calculation of critical expected value (CEV) distributions. The CEV distributions are extensively analyzed.

The GSK statistic is shown to be highly conservative, significantly biased by the presence of zero cells, and less robust as the size of the table increases. The Kullback statistic is shown to be generally liberal with best performance at the lower nominal levels, and little change in robustness characteristics is noted. The Pearson statistic is shown to be somewhat conservative and clearly superior to the other statistics, and the robustness with respect to small expected values is significantly improved as the size of the table increases.

CHAPTER I

INTRODUCTION

The study of cross-classified categorical data in the form of contingency tables has resulted in a wealth of papers, books, and monographs presenting several competing methodologies. These methodologies have been developed through an evolutionary process with contributions from the finest statisticians of the time. Although these methodologies are competing, many of their assumptions, techniques, and problems are similar. Each decade seems to bring a new set of ideas, applications, and discussions related to the use of these methodologies even before previous problems have been solved. The focus of the present study is on the analysis of some of the current methodologies with respect to a classic problem, that of small expected values.

In the past few years there has been a trend toward the analysis of larger and larger contingency tables (more variables and/or more categories per variable). The justification for the use of the current methodologies (which use "approximate" statistics and depend on asymptotic theory) in the analysis of these large tables has been based on large sample sizes. Often, even with large sample sizes, larger contingency tables present some expected cell values which are relatively small. Historically, statisticians have warned against the use of these statistics under the presence of small expected cell values. Yet, there

seems to be little concern for these small values in large tables. The inference is that the problems associated with the use of these statistics become less significant as the size of the table increases.

With the extensive use of the current methodologies, particularly for large contingency tables, there seems to be a pressing need for an investigation of the robustness of these methodologies with respect to small expected values. Of primary interest is the examination of trends as the table size increases.

1.1 Contingency Tables

A contingency table is a presentation of count data resulting from cross-classifications. The cross-classifications are variables, factors, or responses which have a number of levels or categories. Terms used synonymously for this type of data are cross-classified, cross-tabulated, categorical, qualitative, or frequency data. These data are the result of cross-classifying a population, or sample from a population, and accumulating totals for each "cell" of the contingency table. A cell total, then, is the number of observations from the population or sample that fall into the variable, categorical combination represented by that cell. The table summarizes information for the entire population or sample, where every observation is categorized into one and only one cell.

A two-dimensional (two-way), $r \times s$ contingency table has two variables: one variable having r categories and one variable having s categories. The "complete" cross-classification gives a total of $r \cdot s$ cells. The following notation for a two-way, $r \times s$ table will be used:

$\{x_{ij}\} \equiv$ table of observed values;

$\{p_{ij}\} \equiv$ table of cell probabilities;

$\{m_{ij}\} \equiv$ table of expected values;

$$\sum_{j=1}^s x_{ij} = x_{i.} \equiv \text{observed row marginals, } i=1,2,\dots,s;$$

$$\sum_{i=1}^r x_{ij} = x_{.j} \equiv \text{observed column marginals, } j=1,2,\dots,r;$$

$$\sum_{i=1}^r \sum_{j=1}^s x_{ij} = x_{..} = N \equiv \text{total sample size or population.}$$

The marginal probabilities ($p_{i.}, p_{.j}$) and marginal expected values ($m_{i.}, m_{.j}$) are similarly defined. Estimates of the probabilities and expected values will be represented with the usual hat notation (i.e., \hat{p}_{ij} and \hat{m}_{ij}). This notation is easily extended to higher-way tables (tables with more than two variables) simply by adding more subscripts.

Contingency table data is commonly presented in tabular form.

For a 2×3 table, Figure 1 is the usual presentation for the table of

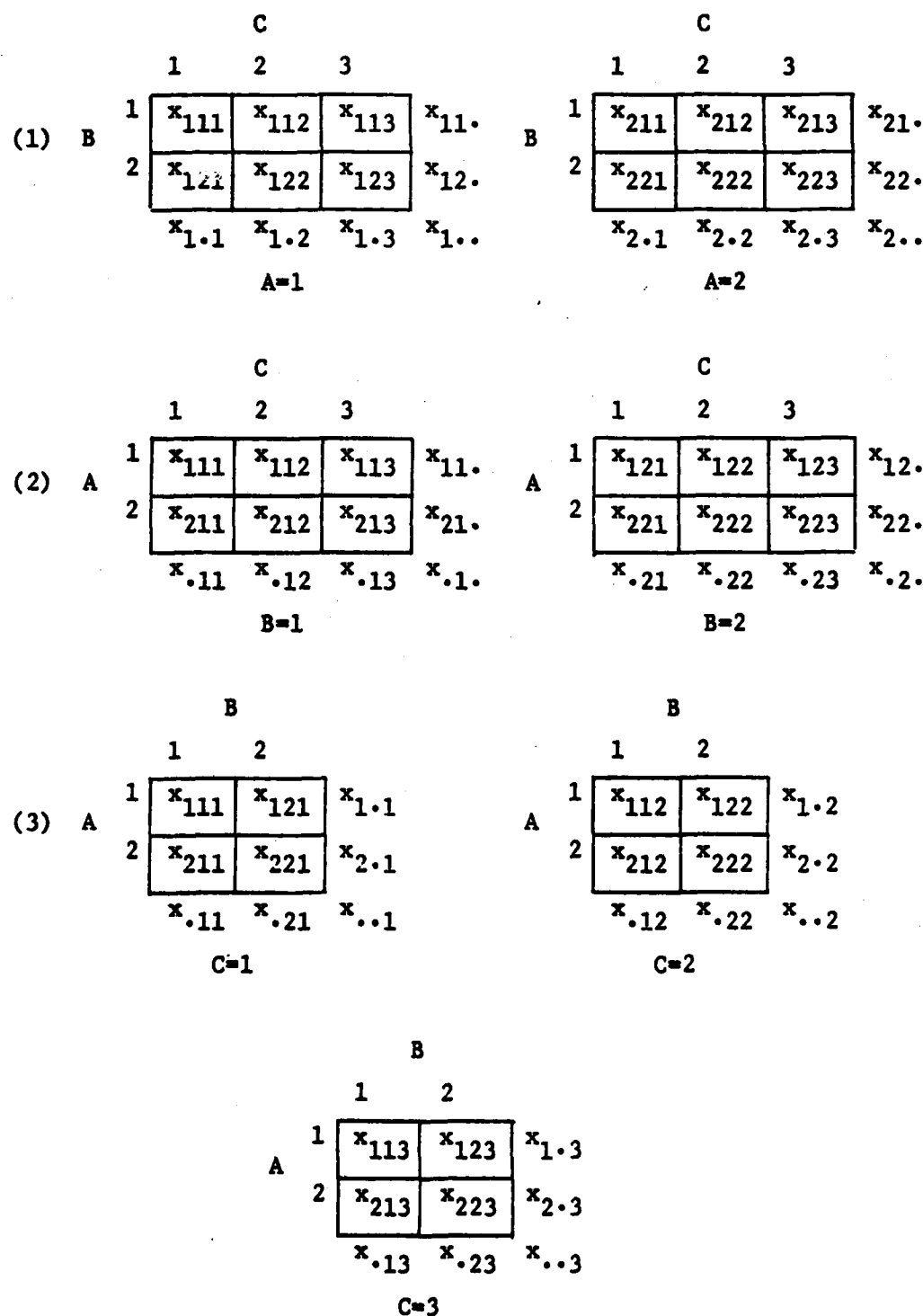
		B			
		1	2	3	
A	1	x_{11}	x_{12}	x_{13}	$x_{1.}$
	2	x_{21}	x_{22}	x_{23}	$x_{2.}$
		$x_{.1}$	$x_{.2}$	$x_{.3}$	N

Figure 1. 2×3 Table of Observations

observations. Here there are two variables: variable A with two categories and variable B with three categories. The marginals and total sample (or population) size are conveniently displayed. Similar tables can be presented for the probabilities and expected values as well as their corresponding estimates. The presentation of the data in tabular form becomes more difficult as the number of variables increases. For example, in the $2 \times 2 \times 3$ table the tabular form can be used with three different orientations as given in Figure 2a. Each orientation allows for easy calculation and presentation of a different combination of sets of "two-way" marginals and a set of "one-way" marginals. Two-way marginals are sums over all variables except two, and one-way marginals are sums over all variables except one. For the three orientations displayed in Figure 2a, the sets of marginals presented are given in Figure 2b.

	<u>one-way marginals</u>	<u>two-way marginals</u>
(1)	$(x_{1..}, x_{2..})$	$(x_{11.}, x_{12.}, x_{21.}, x_{22.})$ $(x_{1.1}, x_{1.2}, x_{1.3}, x_{2.1}, x_{2.2}, x_{2.3})$
(2)	$(x_{.1.}, x_{.2.})$	$(x_{11.}, x_{12.}, x_{21.}, x_{22.})$ $(x_{.11}, x_{.12}, x_{.13}, x_{.21}, x_{.22}, x_{.23})$
(3)	$(x_{..1}, x_{..2})$	$(x_{1.1}, x_{1.2}, x_{1.3}, x_{2.1}, x_{2.2}, x_{2.3})$ $(x_{.11}, x_{.12}, x_{.13}, x_{.21}, x_{.22}, x_{.23})$

Figure 2b. Sets of Marginals for the $2 \times 2 \times 3$ Table

Figure 2a. Orientations for the $2 \times 2 \times 3$ Table

An alternative, and sometimes more convenient, method of presentation of the data from a contingency table is in vector form, using lexicographic order on the subscripts (from left to right). For the 2×3 table the vector of observations would be

$$(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}),$$

and for the $2 \times 2 \times 3$ table the vector would be

$$(x_{111}, x_{112}, x_{113}, x_{121}, x_{122}, x_{123}, x_{211}, x_{212}, x_{213}, x_{221}, x_{222}, x_{223}).$$

This form will be used extensively in this thesis for the presentation of cell probabilities. The term, probability vector, will represent a table of cell probabilities.

1.2 Nature of the Problem

The term, "small expected value", is meaningless without some knowledge of the context of its use. With reference to contingency table analysis, an expected value is "small" if it causes a loss of confidence in the use of a particular methodology. This may depend on many factors, such as the form of the sampling distribution, the assumed model, the hypothesis to be tested, the test statistic, or the nature of the results desired. Unfortunately, for contingency table analysis little theoretical or empirical evidence is available to know when an expected value is really "small". Certain "rules of thumb" and a historical discussion will be presented in Chapter II.

In the analysis of contingency tables the primary interest is to investigate the relationships among the variables. Often, a model is first assumed, then hypothesis tests are performed using statistics and observed data. In classical hypothesis testing situations the null hypothesis is assumed true, and the distribution of a statistic is used to determine whether or not it is reasonable to have obtained the realization of the statistic as calculated from the observed data. In some cases, such as testing the mean of a normal distribution, the form of the distribution is known. In other cases, such as testing the mean of an unknown distribution, certain "large sample" properties of the statistic and an asymptotic distribution are used to test the hypothesis. In still other cases, such as regression analysis and analysis of variance, a model is first assumed. The model is then "fit" based on the observations, and with a distribution assumption, hypothesis tests are performed on the model.

The three situations differ as to the nature of the assumptions and the errors that can occur when there are departures from these assumptions. In the first case, with known distributions there are no assumptions, and the test is exact in the sense that probabilities associated with the hypothesis test errors can be calculated exactly. In the second case, using large sample properties and asymptotic distributions, uncontrolled errors occur because of the approximate nature of the distributions for finite samples. In the performance of these hypothesis tests an understanding of the robustness properties of the associated statistics, with their approximate distributions, is needed, particularly when sample sizes are relatively small. In the third case

uncontrolled errors occur when the assumed distributions are not exact. Here an understanding of the robustness properties of these tests is needed when there are departures from these assumed distributions.

Contingency table analysis incorporates aspects of the second and third cases above. As in the third case a model is usually assumed. In regression analysis and in analysis of variance the model is very significant and is the focus of the procedure. In contingency table analysis hypotheses can be formed without the model, but the model does provide a convenient way of describing the relationships among the variables and the hypothesis tests. Unlike regression analysis and analysis of variance no distribution, related to the model, is assumed. In fact, a "complete" model for a contingency table will be exact. Like the second case above, the distributions of the statistics used to test the hypotheses or model parameters are approximated by their asymptotic distributions. Therefore, as in the second case an understanding of the robustness properties of these tests is needed. The performance of these tests depends on how well the asymptotic distributions fit the exact distributions of the statistics, a criterion dependent on many factors including both sample size and the size of the table.

From the discussion above, the term, "small expected values", in relationship to contingency table analysis refers to the fact that these asymptotic distributions may not well approximate the exact distributions of the test statistics when expected values are small. Hence, robustness properties are related to the performance of these test statistics with respect to small expected values.

1.3 Regression Example

The following regression example provides some insight into the two major aspects of the problem to be investigated in this study. First is the basic problem that these approximate chi-square statistics do not perform well under the presence of very small expected values. Second is the problem that just increasing the size (number of cells) of a table may not improve the performance of the statistics. In fact, there may even be a degradation in performance.

Consider the individual height and weight data given in Table 1.

Table 1. Data for Regression Example

Weight (lbs.)	153	175	104	135	196	177	157	124	145	186
Height (ins.)	70	72	65	68	74	72	70	67	69	73

Assume the following simple linear regression model:

$$y = \alpha + \beta x + \epsilon,$$

where weight (y) is the dependent variable, height (x) is the independent variable, and ϵ is a random variable representing error. Assume that the expected value of the error is zero, that the variance is unknown but the same for each value of x , and that the errors are uncorrelated. Using the least squares procedure to fit the model, the estimate is

$$\hat{y} = -573.27 + 10.26x .$$

The estimate of the correlation coefficient (ρ) is

$$\hat{\rho} = 0.993.$$

With the additional assumption that the errors are normally distributed, a hypothesis test can be performed on the linear relationship of the variables; i.e., $H_0: \beta = 0$ or $\rho = 0$. Under the assumptions and null hypothesis the statistic has a F distribution with one and eight degrees of freedom. At the .01 level of significance $F_{.01,1,8} = 11.26$. The realization of the statistic based on the data is $F_0 = 5732.05$. As might be expected just from the value of the estimate of the correlation coefficient, the variables are highly linearly dependent.

Suppose the data in Table 1 are now categorized and placed into a contingency table, and an independence hypothesis test is performed. First, appropriate categories must be selected. For a 2×2 table let the weight categories be <150 and ≥ 150 and the height categories be <70 and ≥ 70 . The 2×2 table is

		Height		
		<70	≥ 70	
Weight	<150	4	0	4
	≥ 150	0	6	6
		4	6	10

One of the statistics used to test the hypothesis of independence is the Pearson chi-square,

$$\chi^2 = \sum_{i=1}^2 \sum_{j=1}^2 (x_{ij} - m_{ij})^2 / m_{ij},$$

where the expected values (m_{ij}) are estimated under the null hypothesis of independence. In particular, the maximum likelihood estimates are

$$\hat{m}_{ij} = x_{i.} x_{.j} / N; i=1,2; j=1,2.$$

For the table above, these estimates are

1.6	2.4	4
2.4	3.6	6
4	6	10

The value of the Pearson statistic is $\chi^2 = 10.00$. The statistic has an asymptotic chi-squared distribution with one degree of freedom. Rejection values of the statistic corresponding to .10, .05, and .01 levels of significance for the test are 2.71, 3.84, and 6.63, respectively. Therefore, the hypothesis of independence is rejected for as low as the .01 level of significance. This result agrees with the regression test.

Suppose that the size of the table is now increased to a 2×3 table by classifying the heights into the three categories ≤ 68 , (68, 72), and > 72 . The table of observations is

		Height			
		≤ 68	$(68, 72)$	≥ 72	
Weight	< 150	3	1	0	4
	≥ 150	0	2	4	6
		3	3	4	

and the corresponding maximum likelihood estimates are

1.2	1.2	1.6	4
1.8	1.8	2.4	6
3	3	4	

The value of the Pearson statistic is 7.22. The statistic now has an asymptotic chi-squared distribution with two degrees of freedom. The .10, .05, and .01 rejection values are 4.61, 5.99, and 9.21, respectively. The independence hypothesis is no longer rejected at the .01 level of significance, but the hypothesis is still rejected at the .05 and .10 levels.

Increasing the table size to 2×4 using four categories of height, the observed table is

		Height				
		≤ 67	$(67, 71)$	$[71, 73)$	≥ 73	
Weight	< 150	2	2	0	0	4
	≥ 150	0	2	2	2	6
		2	4	2	2	

and the corresponding maximum likelihood estimates are

.8	1.6	.8	.8
1.2	2.4	1.2	1.2

The value of the Pearson statistic is 5.83, and the asymptotic chi-square statistic with three degrees of freedom has .10, .05, and .01 rejection values of 6.25, 7.81, and 11.34, respectively. The independence hypothesis cannot be rejected at even the .10 level. There appears to be a trend as the size (number of cells) of the table increases.

Before discussing this "apparent" trend, consider the 3×3 table where the weight variable is divided into three categories. The observed table is

		Height			
		<u><68</u>	(68,72)	<u>>72</u>	
Weight	<u><145</u>	3	1	0	4
	(145,175]	0	2	1	3
	>175	0	0	3	3
		3	3	4	

and the corresponding maximum likelihood estimates are

1.2	1.2	1.6	4
.9	.9	1.2	3
.9	.9	1.2	3
3	3	4	

The value of the Pearson statistic is 11.11 and can be compared against four degree of freedom values of the chi-squares statistic at the .10, .04, and .01 levels of 7.78, 9.49, and 13.28, respectively. The observed value of the statistic again calls for a rejection of the null hypothesis of independence at the .10 and .05 levels but not at the .01 level.

Table 2 summarizes the four contingency table tests above, along with an additional test based on the most extreme table. The most extreme table is formed by defining a separate category for every observed height and weight. For the data in Table 1, this results in a 10×8 table. The Pearson statistic can easily be calculated with a formula given by Cramér (1946, pg. 443),

$$X^2/N(q-1) \leq 1,$$

where $q = \min(r,s)$ for an $r \times s$ table. The value of one is obtained when all the rows or columns contain one and only one entry different from zero. Then,

$$X^2 = N(q-1). \quad (1-1)$$

Table 2. Contingency Table Tests of Regression Example

Table	(k) No. Cells	$\bar{m} = N/k$	Min \hat{m}_{ij}	Pearson X^2	d.f.	Chi-Square Distribution .10 .05 .01	Hypothesis Tests
2 x 2	4	2.50	1.6(1)	10.00	1	2.71 3.84 6.63	Reject H_0 at all α
2 x 3	6	1.67	1.2(2)	7.22	2	4.61 5.99 9.21	Reject H_0 at .10, .05
2 x 4	8	1.25	.8(3)	5.83	3	6.25 7.81 11.34	Do not reject H_0 at any α
3 x 3	9	1.11	.9(4)	11.11	4	7.78 9.49 13.28	Reject H_0 at .10, .05
10 x 8	80	0.13	.1(60)	70.00	63	78 82 92	Do not reject H_0 at any α

Note that this criterion is met in the 2×2 table, and therefore, $\chi^2 = 10(2-1) = 10$. For the extreme 10×8 table, $\chi^2 = 10(8-1) = 70$.

Referring to Table 2, the trend from the 2×2 through the 2×4 tables is that the χ^2 statistic is decreasing and becoming less sensitive to departures from the null independence hypothesis. The consideration of the 3×3 table reverses this trend since the χ^2 statistic increases and again rejects the hypothesis at the .10 and .05 levels. At the extreme 10×8 table the statistic has become very large, 70; but, the hypothesis is not rejected at any of three levels. What is interesting about this example is that the data remain constant, and the variables, based on the regression test, seem so clearly dependent. Why are there such discrepancies in these hypothesis tests? The answer lies in the nature of the χ^2 statistic and the error involved in using the chi-squared distribution as an approximation to the exact distribution of the statistic.

The χ^2 statistic has an exact, discrete distribution. For a given sample size and given size table there are only a discrete number of possible observed tables. The chi-squared distribution is continuous, and therefore some error is always present. For a two-way, $r \times s$ contingency table the number of degrees of freedom associated with the independence hypothesis test is $(r-1)(s-1)$. Basically, the number of degrees of freedom of the test increases as the size of the table increases. The power of a test is a measure of the sensitivity of the test to departures from the null hypothesis. Normally, in hypothesis testing situations the power of the test improves as the degrees of freedom increase. This is not the case in the example above. In fact,

just the opposite is true from the 2×2 through the 2×4 tables. With the fixed data, as the size of the table increases, the table becomes sparser in the sense that the average expected value ($\bar{m} = N/k$, where k is the number of cells) decreases. Table 2 lists these average expected values and the minimum expected values for each table of the example. From the first three tables it might be concluded that the power of the test decreases as the table becomes sparser, but the 3×3 table would invalidate this conclusion. Note, however, that the minimum cell expectation (MCE) decreases from the 2×2 to the 2×4 table and then increases with the 3×3 table. Hence, it might be concluded that the power decreases as the MCE decreases. Whatever the case, the power of the test is clearly affected by the cell expected values. The reason for this is that the chi-squared distribution is not performing well as an approximation for the exact distribution of the X^2 statistic under the presence of small expected values.

Before leaving this example, two additional comments should be made. The first is that the selection of the boundary points for the categorization of the data was critical to the results of the hypothesis tests. The points were deliberately chosen to demonstrate certain aspects of the small expected value problem. Selection of other boundary points could lead to other hypothesis test decisions. However, the extreme table is unique. Here the hypothesis could not be rejected at any of the three levels of significance. It is interesting to note that as long as one of the variables has unique data points, the Pearson X^2 statistic can be calculated from Equation (1-1), and it will never reject the null hypothesis at the .10 level regardless of the data,

sample size, or size of table. The optimum selection of these boundary points is another problem [e.g., see Hamdan (1968)]. In most contingency table applications the data are not the continuous-type data as in the previous example. Usually the data are true count data, resulting from the cross-classification of a sample or population, and an extreme table does not exist.

The second comment is that the power of this test can, in general, be improved by increasing the sample size without changing the size of the table. This would not change the degrees of freedom of the test, but would increase the average expected values and, in general, increase the minimum expected values. The chi-squared distribution would better approximate the exact distribution of the statistic, and the power of the test would, in general, increase. In the previous example suppose the same data were exactly replicated. The regression test would be improved (made more powerful) since its "degrees of freedom" would now be 18 for the F-test of independence. However, while the X^2 values in Table 2 would double, the degrees of freedom of the chi-square tests would not change. These tests would now all reject the hypothesis of independence at all levels of significance. The minimum expected values and average expected values would all double. The power of the tests would clearly increase.

This example demonstrates several significant aspects of contingency table tests. The power of these tests is related to how well the chi-squared distribution approximates the exact distribution of the statistic and to the size of the expected values. For a fixed sample size these expected values decrease as the size of the table increases,

and in spite of the increase of the degrees of freedom of the test, the power can still decrease. In general, as the sample size increases the power of the test increases because, with an increase in the expected values, the chi-squared distribution better approximates the exact distribution of the statistic. Finally, the discrete nature of the statistic causes perturbations, making it difficult to observe any specific trends.

1.4 Scope

This study focuses on the most popular statistics currently in use to test independence hypotheses in contingency tables. The primary purpose of this study is to investigate the robustness characteristics of these statistics with respect to small expected values as the size of the table increases. A secondary purpose is to compare the performance of these statistics and make recommendations for their use. The main vehicle for the study is a Monte Carlo simulation. The simulation provides estimates of exact levels of significance to compare with specified nominal levels. Trends are discussed and conclusions drawn with respect to the stated purposes.

Chapter II presents a historical overview of the development of contingency table analysis, the current statistics, and the small expected value problem. Also, previous numerical and empirical studies are extensively reviewed. A convenient table summarizes these studies, and a final subsection discusses their limitations and restrictive scope with respect to the small expected value problem.

Chapter III provides a detailed presentation of the current methodologies for the analysis of contingency tables. This chapter begins

with a discussion of the underlying sampling models and the assumed log-linear model and relates them to the independence hypotheses. Maximum likelihood estimates are then derived. Next, the three current, competing statistics are presented. Finally, some comparisons are made showing the asymptotic equivalence of these statistics.

Chapter IV describes the experimental design. This includes discussions of the selection of parameters and the designs of the Monte Carlo and exact programs. Here, justifications are given for the selection of the primary level of significance parameter, the probability design parameters, the sample sizes, and the number of tables to be generated by the Monte Carlo simulation. The problem of zero cells and zero marginals is also discussed. The last subsection provides a validation of the Monte Carlo simulation using exact results.

Chapter V presents the major results of the study through the analysis of the data. The chapter begins with a presentation and analysis of the basic data from the Monte Carlo simulation and the exact algorithm. Comparisons are made with results from other studies. Following a procedure used in a previous study, minimum and maximum significance levels for minimum cell expectation intervals (MCEI) are reported and analyzed. Minimum sample sizes are then derived for a specified criterion, and critical expected value (CEV) distributions are calculated. These CEV distributions are extensively analyzed and trends noted.

Chapter VI lists the significant conclusions of the study, provides recommendations for the use of the statistics investigated, and makes recommendations for future research.

CHAPTER II

HISTORICAL OVERVIEW AND LITERATURE REVIEW

This chapter presents a historical overview of contingency table analysis with particular emphasis on the current methodologies to be investigated and the small expected value problem. Also, previous studies related to this problem are extensively reviewed.

2.1 Methodologies

Methodologies for the analysis of contingency tables have, to varying degrees, differed in several ways:

- (1) sampling model
- (2) method of estimation of expected values
- (3) assumed model of the parameters
- (4) method for selecting the appropriate model
- (5) interpretation of the model selected
- (6) test statistics
- (7) considerations of "factors" and "responses".

A review of each of these topics will not be undertaken. This review will concentrate on (2), (3), (4), and (6) which lead to the specific differences of the most popular competing methodologies today.

Most statisticians agree [e.g., see Cochran (1952) and Bishop, Fienberg, and Holland (1975)] that the classic paper by Pearson (1900b) was the first to propose a useful statistic, X^2 (Pearson's chi-square

goodness-of-fit), to use in the analysis of contingency tables. In this paper Pearson established the necessary distribution theory for finding significance levels when expected values can be calculated under a null hypothesis. The X^2 statistic,

$$X^2 = \sum_i (x_i - m_i)^2 / m_i,$$

compares the observed values (x_i) with these expected values (m_i) based on the chi-squared distribution. Fisher (1924) outlined a proof of this limiting chi-squared distribution for X^2 when the parameters are estimated from the observations. A complete proof was given by Cramér (1946) and later, under more general conditions, by Birch (1964). Pearson (1900a) used the test specifically for a 2×2 table and later [Pearson (1904)] for an $r \times s$ table.

Pearson (1900b) assumed an underlying multivariate normal distribution for the data in all cases, even when the data was from a discrete source such as categorical variables. In contrast, Yule (1900) proposed considering the levels (categories) of the variables as fixed. He investigated the relationship of these variables by considering a cross-product ratio for the 2×2 table,

$$\alpha = \frac{m_{11}m_{22}}{m_{12}m_{21}},$$

where the m_{ij} are the expected values. He looked at several functions of this cross-product ratio, using them to compare various tables with known structures. For the next 25 years an argument between Yule and

Pearson over these competing approaches dominated the literature on contingency tables [e.g., see Pearson and Heron (1913) and Fisher (1922a,1924)]. Pearson's goodness-of-fit statistic, X^2 , has a multitude of applications [see Lancaster (1969a)] and is one of the most used statistics today. Pearson's underlying continuous model for categorical data has been essentially disregarded, except for some use by Lancaster (1949, 1950, 1957, 1969a) in a partitioning of chi-square method. Yule's concept of considering the categories as fixed, with some modifications, is the historical beginning of today's categorical data models.

Both Yule and Pearson considered only two-dimensional (variable) structures. It was not until Bartlett (1935) that a higher-way (three or more variables) table was investigated. Bartlett analyzed a $2 \times 2 \times 2$ table and extended Yule's cross-product ratio to define the concept of second-order interaction. He also proposed a chi-square test for the presence of this interaction based on the following sum of squares statistic:

$$\sum_{r=1}^8 x^2/a_r = x^2 \sum_{r=1}^8 1/a_r,$$

where x is the total deviation from the expectation (a_r) in each cell. The similarity to Pearson's X^2 is obvious. This classic, concise paper also presented several new ideas: it linked the term "independence" to "no interaction", offered an exact test when one of the cell expected values was small, suggested an extension of the procedure to 2^k tables,

and demonstrated the problems associated with multi-way tables having variables with more than two levels (categories).

Roy and Kastenbaum (1956) further extended Yule's theory and Bartlett's higher-order interactions and tests. They introduced a "multiplicative" model which generalized Bartlett's second-order interaction (which they called "three-factor interaction") from $2 \times 2 \times 2$ tables to $r \times s \times t$ tables. Birch (1963) showed that these interactions could be defined as linear combinations of the logarithms of the expected frequencies. He introduced a special parameterization and demonstrated the relationship between the values of these parameters and the correspondence of the observed and expected values. This formed the basis for what is known as the log-linear model. This model provides the flexibility to analyze the structure of any contingency table, in a manner similar to analysis of variance and stepwise regression, using a number of different methodologies. The log-linear model has been extensively described by Bishop, Fienberg, and Holland (1975) with supporting mathematical proofs given by Haberman (1970, 1973, 1974) and contributions in the use of the model by Bishop (1967, 1969a,b), Goodman (1968, 1970, 1971a,b, 1973), Fienberg (1968, 1970a,b, 1977, 1979), Grizzle and Williams (1972), and many others.

Other models have been suggested, including an additive model [Bhapkar and Koch (1968)], the Lancaster (1949, 1950, 1969a) partitioning model, and a general linear model [Nelder and Wedderburn (1972) and Nelder (1974)] with the log-linear model as a special case. The additive model has been used in special situations such as sample survey data, drug comparisons, and biological assays [e.g., see Johnson and

Koch (1970) and Koch and Reinfurt (1971)]. Johnson and Koch (1970) discuss the advantage of the additive model for sample survey data. In general, the log-linear model is the most extensively used, providing convenient parameters for most hypothesis testing situations. An excellent discussion and comparison of the corresponding additive and multiplicative interaction terms for these two models is given by Darroch (1974).

The current, competing methodologies use the log-linear model, and primarily differ in the method of estimating expected values and in the test statistics used to select appropriate parameters for the log-linear model (hypothesis testing). The three most popular methods of estimation are maximum likelihood, minimum modified chi-square, and minimum discrimination information. The four most popular test statistics are the Pearson X^2 , the minimum modified chi-square, the minimum discrimination information, and the likelihood (log-likelihood) ratio.

Maximum likelihood estimates are based on the well-known theory presented by Fisher. Although he makes no reference to maximum likelihood, Bartlett (1935) used these estimates for his expected values ($a_{r\cdot}$) in a $2 \times 2 \times 2$ contingency table. Bartlett provided a cubic equation, given to him by Fisher, to solve for these expected values. The equation cannot be solved in closed form, but only through trial and error or some numerical iteration technique. It wasn't until Deming and Stephen (1940) that a general iterative scheme was presented to solve for the maximum likelihood estimates in a three-way table. Brown (1959) gave a first proof, although somewhat incomplete, on the convergence of the procedure. Ireland and Kullback (1968) offered a complete

proof. Other proofs were given by Fienberg (1970b), Darroch and Ratcliff (1972), and Haberman (1970, 1973, 1974a). Haberman (1972) presented an excellent computer program of the procedure for a general N-way table. As contingency table analysis evolved, Roy and Kastenbaum (1956), Lewis (1962), Birch (1963), Goodman (1963, 1964, 1968, 1971a,b, 1973), Bishop (1967, 1969), and others have used these estimates.

Maximum likelihood estimates were well-known to have desirable asymptotic properties such as consistency and efficiency. Neyman (1949), in a classic paper, defined a class of estimates, best asymptotically normal (BAN), which included the maximum likelihood estimates. He discussed their properties in detail. Neyman (1949) went on to specifically define other BAN estimators, in particular, a minimum modified chi-square estimator using the generalized quadratic theory established by Wald (1943). Along with these estimates, Neyman proposed a corresponding test statistic similar to the Pearson χ^2 ,

$$Y^2 = \sum_i (x_i - m_i)^2 / x_i,$$

where the observed values now divide the squared differences of the observed and expected values. Bhapkar (1966) demonstrated the equivalence of this statistic and the Wald statistic for linear hypotheses and, with a linearization technique, for nonlinear hypotheses. Grizzle, Starmer, and Koch (1969) generalized this theory and established a procedure (GSK) based on weighted least squares. The GSK procedure has been applied to a series of different kinds of contingency tables using linear and log-linear models [e.g., Koch and Reinfurt (1971),

Johnson and Koch (1971), and Grizzle and Williams (1972)]. Today, due to the extensive and varied use of the GSK procedure and the availability of excellent computer programs [e.g., Landis et.al. (1976)], this technique has become very popular.

The third approach is closely related to maximum likelihood and is based on the theory of minimum discrimination information (MDI) as presented by Kullback (1959). This theory was first applied to contingency tables by Kullback, Kupperman, and Ku (1962). Later applications and expansion of the theory were given by Ireland and Kullback (1968), Ku and Kullback (1968, 1974), Ireland, Ku, and Kullback (1969), Ku, Varner, and Kullback (1971), Gokhale (1972), Berkson (1972), and many others. The theory and procedures, together with excellent examples and computer programs, have been presented in a book by Gokhale and Kullback (1978a). Estimates are obtained using iterative techniques [see Ku and Kullback (1968)] in a manner similar to those for maximum likelihood estimates. In fact, for independence hypotheses these estimates are identical to the maximum likelihood estimates [see Gokhale and Kullback (1978a, pg. 3) and Meyer (1980)]. For independence hypotheses the test statistic is the minimum discrimination information statistic (MDIS),

$$2I = 2 \sum_i x_i \ln(x_i/m_i),$$

where the above MDI estimates for the expected values (m_i) provide the minimum value for this statistic. This statistic is identical to the log-likelihood ratio statistic first proposed by Fisher (1922b), later

used for independence in contingency tables by Neyman and Pearson (1928), and extended to other contingency table hypotheses by Wilkes (1935), Woolf (1957), Chakravarti and Rao (1959), and Gart (1966). For other hypothesis tests, such as homogeneity, the MDI estimates are not maximum likelihood, and the MDIS is not log-likelihood ratio [e.g., see Gokhale and Kullback (1978b)]. Probably the most interesting aspect of the MDI/MDIS approach is that, as first shown by Kullback (1959), the log-linear model is a natural result of the theoretical derivation of the minimization of the statistic.

All the methods previously discussed have been approximate, dependent on asymptotic distributions and requiring data to be "relatively" large. Fisher (1934) first proposed an "exact" test for 2×2 tables. Actually, Yates (1934) first published the test but gave recognition to Fisher. Irwin (1935) independently presented the same procedure. The Fisher (Fisher-Irwin or Fisher-Yates) exact test considers the margins of the observed table as fixed and calculates the hypergeometric probability of the observed table and all tables farther from the "no interaction" hypothesis. The total sum of these probabilities is then compared to a given level of significance. Freeman and Halton (1951) applied the test to a general $r \times s$ table, giving appropriate formulas. Computer programs for $r \times s$ tables were developed by March (1972) and Schwartz (1972). Shaffer (1972) presented a technique to extend the Fisher test to higher-way (three or more variables) tables. Other similar "exact" tests have been suggested by Halpern (1937, 1940), Finney (1948), Tocher (1950), Leslie (1955), Boschloo (1970), Zahn and Roberts (1971), and Agresti and Wackerly (1977). In Fisher and Irwin's

time, due to the calculations required, the exact test was only useful for very small tables ($N \leq 20$). Even today, with the availability of high-speed computers the usefulness of the test is restricted. Guidelines for use of the exact test for $r \times s$ tables on an IBM 360 computer have been given by Agresti, Wackerly, and Boyett (1979), and a more efficient algorithm has recently been presented by Pagano and Halvorsen (1981). Exact procedures are no panacea. They have frequently been attacked simply on the basis of the assumption of fixed margins [e.g., see Camilli and Hopkins (1978) or Berkson (1978)]. Due to their computational difficulties, these procedures have seen little use in practice. They have primarily been used to validate or compare the approximate methods.

2.2 Small Expected Values

All three of the approximate methods discussed are based on the BAN estimators described by Neyman (1949) and are asymptotically equivalent. The asymptotic distribution for each statistic is the chi-squared. The inaccuracy of chi-square tests for small samples was well-known to Pearson, Fisher, Yule, and Yates in the early development of contingency table analysis. In practice, as stated by Yates (1934), it was customary to regard the Pearson chi-square tests as "sufficiently" accurate if no cell had an expected value less than five. No empirical or theoretical evidence had been presented to verify this magic "rule of thumb". Even today, most statistical textbooks give the five "rule of thumb" for chi-square tests for any goodness-of-fit or contingency table hypothesis. However, a brief survey of the literature reveals as many as nine other "rules of thumb".

Cochran (1952) presented a detailed discussion of the development of the chi-square goodness-of-fit test (Pearson's X^2) along with his classic summary of recommendations for the use of Fisher's exact test, Pearson's X^2 test, Haldane's (1937, 1940) exact mean and variance calculations, and Yates' (1934) continuity correction to Pearson's X^2 . These recommendations were based, as Cochran states, more on "experience" than anything else. They did provide, at the very least, a "straw-man" for statisticians to evaluate. With the exception of the five "rule of thumb", these guidelines have been discarded due to problems associated with Yates' continuity correction [e.g., see Conover (1974) or Haber (1980)], current use of the log-linear model and associated statistical techniques, and the availability of high-speed computers.

Some investigations have been made with regard to the properties of current procedures when expected values are small. Most of these have been empirical, either numerical comparisons or Monte Carlo studies. Haberman (1977) presented the only recent theoretical proofs providing some insight into the asymptotic properties of statistics for large contingency tables with small expected frequencies. His results are related to the log-linear model. Conditions are provided for the asymptotic normality of linear functionals of log-mean vectors (maximum likelihood estimates) and for the asymptotic chi-squared distribution of the Pearson X^2 and the log-likelihood ratio statistics. Essentially, these conditions will allow some cells to have fixed small expected values while asymptotic properties are maintained. These results hold for any size table. Apart from this, little theoretical information is avail-

able concerning the effect of small expected values on large tables.

2.3 Review of Previous Studies

Most of the empirical work related to chi-square tests have focused on the Pearson X^2 as a goodness-of-fit test for the multinomial. Of particular note is Yarnold's (1970) extensive investigation which provided significant insight into the small expected value properties of this test. The first empirical studies related to contingency tables, although somewhat limited in scope, provided some interesting results and ideas for follow-on studies. The studies of Odoroff (1970) and Craddock and Flood (1970) are the most frequently discussed. However, several other studies appeared earlier, including Monte Carlo studies by Lewontin and Felsenstein (1965) and Craddock (1966), an "exact" study by Hayman and Leone (1965), and a numerical comparison study by Sugiura and Otake (1968). These studies along with follow-on studies will be reviewed in approximate chronological order. A summary chart is provided in Table 3 and a discussion of the limitations of these studies is presented in subsection 2.3.16. Only one of the studies [Camilli and Hopkins (1978)] that is related solely to continuity corrections in 2×2 tables will be reviewed. The results of this study are representative of the results of other similar studies [e.g., Plackett (1964), Grizzle (1967), Mantel and Greenhouse (1968) and Conover (1974)]. In a recent study Haber (1980) has provided some different conclusions concerning "alternative" (to Yates) continuity corrections.

2.3.1 Lewontin and Felsenstein (1965)

Lewontin and Felsenstein presented a Monte Carlo investigation of the Pearson X^2 test of homogeneity in $2 \times N$ tables. They compared

the estimated exact distributions of the Pearson X^2 statistic with the chi-squared distribution in the tail areas, specifically at certain nominal values (.01, .02, .05, .10). They considered 27 different combinations of fixed marginals and probability arrangements for 2×5 and 2×10 tables. Only symmetric N-column marginals with variances of 0, 1.6, and 3.6 were considered. They classified each table arrangement as to how well the distribution of the Pearson X^2 statistic agreed with the chi-squared distribution.

The results of the study indicated that the Pearson X^2 statistic was generally conservative for five or more degrees of freedom and more robust with respect to small expected values than usual "rule of thumb" would indicate. Lewontin and Felsenstein suggested that the statistic could be used as a test of homogeneity in $2 \times N$ tables when expected values were as low as one. It should be noted that the sampling plan used for their Monte Carlo procedure restricted both sets of marginals, which implies a hypergeometric sampling model rather than the product-binomial model related to the homogeneity hypothesis (see Section 3.1). This raises some doubt as to the general validity of their conclusions.

An interesting aspect of this study was the calculation of estimated mean and variances for the X^2 statistic in all 54 cases considered. The agreement with exact formulas given by Haldane (1940) was excellent.

2.3.2 Haynam and Leone (1965)

Haynam and Leone studied the Pearson X^2 statistic when used both as a goodness-of-fit test statistic for the multinomial and as an independence and homogeneity test statistic for contingency tables. With regards to contingency tables their study investigated the exact dis-

tributions of X^2 under the three sampling conditions of no margins fixed (multinomial), one set of margins fixed (product-multinomial), and both sets of margins fixed (hypergeometric) in 2×3 and 3×3 tables with equiprobable cells. They compared these exact distributions with the chi-squared distribution.

Results were similar for all three sampling models. Haynam and Leone concluded that the distribution of the X^2 statistic is fairly well approximated by the χ^2 distribution in the region up to 0.9 cumulative probability, but tended to overestimate "rather badly" exact probabilities in the tail area. Unfortunately, for hypothesis testing situations the tail areas are of primary importance.

This study was very limited by the consideration of only the equiprobable case and sample sizes of ten and fifteen. However, as the authors mention, the amount of computer time needed for these exact calculations was significant.

2.3.3 Craddock (1966)

Craddock performed a Monte Carlo investigation of the Pearson X^2 statistic when used in the 3×3 equiprobable contingency table to measure the "evidence of association" (test for independence). Craddock generated 10,000 observations of tables for sample sizes (N) of 10, 15, 20, 25, 30, 40, 50, and 100. He then calculated the Pearson X^2 for each table and formed histograms in the range 0 to 20 using a grouping interval of 0.2. He reported estimated percentile values for these histograms for the important percentiles of 50, 90, 95, 98, 99, and 99.9 and compared these to the chi-squared distribution values.

Craddock's results indicated excellent agreement to the chi-

squared distribution throughout the entire range from $N=100$ down to $N=30$. For lower values of N the agreement was "spotty"; but, Craddock suggested that at the important hypothesis testing percentiles the Pearson statistic could be used with these 3×3 tables for N as low as 10.

2.3.4 Sugiura and Otake (1968)

Sugiura and Otake used numerical evaluations to compare various "improved" tests for $r \times s$ contingency tables with small frequencies in some cells. The test statistics considered were:

- (1) Fisher-Irwin exact test
- (2) Pearson X^2 test (maximum likelihood estimates)
- (3) Yates' continuity correction
- (4) Likelihood Ratio (LR) test
- (5) LR test with Dandekar modification [Rao (1952, pg. 203)]
- (6) LR test with Yoshimura (1963) correction
- (7) Pearson X^2 test with authors' correction
- (8) Method I by Nass (1959)
- (9) Method II by Nass (1959)
- (10) Method I by Gart (1966)
- (11) Method II by Gart (1966)
- (12) Modified Dandekar method by authors.

The authors used four examples for the study: a 2×2 example from Rao (1952), a 2×2 symmetric case, a 2×2 large marginals case, and a 2×3 example from Yates (1934). They used the exact test (1) as their criterion for the 2×2 tables and, for the 2×3 table, exact tests

based on ordering of tables by the Pearson X^2 statistic and the LR statistic.

Sugiura and Otake's study indicated that the methods (5), (10), (11), and (12) were somewhat better than the others for 2×2 tables, though they were more conservative than exact levels. Gart's methods (10) and (11) were conservative for 2×3 tables. The LR modification by Yoshimura (6) always performed better than the LR statistic (4). Nass's methods (8) and (9) did not perform very well. Other methods gave results similar to (2).

It should be noted that most of these tests were restricted to two-way tables, and some required one variable restricted to two categories. The results of this study are very limited. Only four examples were considered, and the exact test itself is known to be conservative and may not be a good criterion for comparing other tests.

2.3.5 Odoroff (1970)

Odoroff presented a study to compare the small sample properties of twelve tests for interaction in $2 \times 2 \times 2$ and $3 \times 2 \times 2$ contingency tables. The twelve tests were formed by combining three test statistics (Pearson X^2 , minimum logit chi-square, and likelihood ratio) with four estimation procedures [iterative maximum likelihood and three variations of minimum logit chi-square due to Gart and Zweifel (1967)]. One set of two-way marginals were fixed, and the observations were considered to be sampled from $r \times s$ independent binomial distributions. The results were thus limited to $r \times s \times 2$ tables under this product-binomial sampling scheme. Odoroff claimed that his hypothesis formulation was equivalent to Bartlett's (1935) no second-order interaction. An evaluation was made

as to the adequacy of the chi-squared distribution in approximating exact levels of significance (nominal levels at .05 and .01) under the null hypothesis. Also, power calculations were performed under selected alternatives and comparisons made as to computational simplicity. For the $2 \times 2 \times 2$ tables calculations were made by complete enumeration, so levels were exact. For $3 \times 2 \times 2$ tables a Monte Carlo method was used to provide estimates of exact levels.

Odoroff generally concluded that the exact levels of the minimum logit chi-square tests and the Pearson X^2 test were better approximated by the chi-squared distribution than were the exact levels of the likelihood ratio. The exact levels of tests using the minimum logit chi-square estimation were better approximated than the exact levels based on maximum likelihood estimation. No significant power differences were observed. The non-iterative minimum logit chi-square estimations, the corresponding test statistic, and the Pearson X^2 were the easiest to compute.

An interesting aspect of Odoroff's study was that results were presented in relation to a special computation of minimum cell expectation (MCE). Odoroff used these MCE to control his selection of cell probabilities. In general, the statistics performed worse for lower MCE.

2.3.6 Craddock and Flood (1970)

Craddock and Flood made a Monte Carlo investigation of the Pearson X^2 statistic in $r \times s$ contingency tables from 3×2 to 5×5 with expected values under the null hypothesis of independence from one to five. The study considered only equiprobable tables, so the expected values were equal for all cells.

Craddock and Flood developed histograms for the calculated Pearson X^2 and provided "smoothed" tables of percentile values ranging from one to 99.9 percent. They demonstrated the convergence of the distribution of the Pearson X^2 statistic to the chi-squared distribution throughout its range. Focusing on the 5% and 50% percentiles, they provided curves to reflect speeds of convergence. They concluded that the normal five "rule of thumb" for minimum expected values, at least for these equiprobable tables, was too conservative. From the smoothed histograms they observed that the behavior of the Pearson X^2 statistic for a given value of N was in general better for larger tables. This led them to suggest that with a fixed sample size analysts should consider using larger contingency tables even if expected values are below five.

2.3.7 March (1970)

March performed a study of the Pearson X^2 statistic in 2×3 tables. He considered all possible tables for sample sizes (N) of six through 30 and 36 and 42 where the expected value in each cell is greater than or equal to one. He compared the chi-square probabilities of the X^2 statistic [i.e., $P(\chi^2 \geq X^2)$] and the exact cumulative hypergeometric probabilities of the 2×3 table with fixed marginals. These probabilities were compared in terms of their absolute percentage differences and on their extent of agreement in certain specified exact probability regions, (.000-.005), (.005-.010), (.010-.050), (.050-.100), (.100-.150), (.150-.200), and (.200-1.00). Also, the number of times that the chi-squared distribution underestimated and overestimated the exact probabilities were calculated.

March's results showed that the Pearson X^2 approximation improves as the sample size (N) increases and, hence, as the average expected value (N/6) increases. As a general approximation to the exact hypergeometric distribution, March rated the X^2 statistic as poor. However, March noted that when used as a test of significance (independence) with the usual nominal levels (e.g., .05), the X^2 statistic gave results very similar to the "exact" test even for relatively low expected values (i.e., less than five).

An interesting aspect of March's study was the running accumulation of overestimates and underestimates. March generalized that the ratio of underestimates to overestimates increased as the exact probability increased and decreased as the sample size decreased.

2.3.8 Yarnold (1970)

Yarnold came to similar conclusion as Craddock and Flood and others concerning the conservative nature of the five "rule of thumb" for the Pearson X^2 statistic. He limited his empirical investigation to the goodness-of-fit hypothesis of a multinomial distribution.

As a result of his investigation, Yarnold provided a rule for guiding the use of the Pearson X^2 statistic in any general hypothesis testing situation where it is appropriate. Yarnold states that, "If a number of classes s is three or more and if r denotes the number of expectations less than five, then the minimum expectation may be as small as $5r/s$." This is essentially an extension of Cochran's (1952) recommendations and broadens the use of the statistic to include conditions when some expected values are less than five. It remains to be seen if the rule is appropriate for contingency table tests.

2.3.9 Roscoe and Byars (1971)

Roscoe and Byars performed a Monte Carlo study of the Pearson χ^2 test for goodness-of-fit and for "independence" in a "typical behavior research situation". Only the later test will be discussed here. The "independence" test was based on sampling for the $r \times s$ tables with the independent row marginals fixed. Thus the test was actually the test for homogeneity across rows (see Section 3.3). Tables varied from 2×2 to 5×5 . Three types of underlying sampling distributions were used: uniform, moderately skewed, and extremely skewed. Fixed sample sizes for the independent rows were equal, although the authors did comment on an extension to the reported study allowing for unequal sample sizes.

Roscoe and Byars demonstrated that under the null hypothesis of homogeneity when sampling from a uniform distribution, "reasonable" results [using Cochran's (1952) criteria for exact levels of significance as compared to nominal levels] are obtained at the .05 nominal level with average expected values as low as two and at the .01 level with values as low as four. Their results further indicated that with moderate departures from uniformity expected values should be maintained at four for the .05 level and six for the .01 level, and with extreme departures these values should be six and ten, respectively.

It should be noted that these recommendations are for "average" expected values, and under departures from uniformity they allow for some relatively small expected values to be balanced by larger ones. Roscoe and Byars believed that this approach would be most helpful since most studies involve a uniform or equiprobable distribution,

and comparisons could be made.

2.3.10 Margolin and Light (1974)

Margolin and Light investigated the small sample behavior of Pearson's X^2 , Kullback's minimum discrimination information statistic (2I), a sample version of Goodman and Kruskal's lambda (L), and their own C statistic. They considered a 3×2 table under the hypothesis of homogeneity where the marginals $x_{.1}$ and $x_{.2}$ are fixed, and limited their examination to $x_{.1} = x_{.2} = N/2$, N taking values of five and 10. They used 11 probability structures for the marginal $p_{i.}$ ($i=1,2,3$). They calculated exact tail area probabilities (levels of significance) for each statistic at nominal levels of .10, .05, .025, and .01 by completely enumerating the tables for each structure and using the chi-squared distribution as the null distribution. A very interesting aspect of this study included separately considering the case where marginal observed $x_{i.}$ were zero. Margolin and Light used the chi-squared distribution with one degree of freedom instead of two for these situations. This provided both exact "conditional" and "unconditional" tail area probabilities.

In general, the results of this study indicated that in these tail areas the null distribution of C was better approximated by the chi-squared than the null distribution of X^2 , and both were considerably better approximated than the null distribution of 2I. L was so erratic and liberal that it was clearly not appropriate for use as a test statistic for homogeneity. Margolin and Light's results further indicated that the C and X^2 statistics were well approximated at or near the equiprobable vector. For other vectors the X^2 statistic was

generally conservative and the C statistic showed no pattern. The three statistics (X^2 , C, 2I) performed much better under the "conditional" criterion. This should have been expected since there is indeed a loss of degrees of freedom under the presence of zero marginals, as has been shown by many authors [e.g., see Bishop, Fienberg, and Holland (1975, pp. 115-119)].

Margolin and Light concluded their study with a theorem and proof relating the X^2 and 2I statistics in $r \times 2$ tables where $x_{.1} = x_{.2} = N/2$. The relation is

$$(2 \ln 2)X^2 \geq 2I \geq X^2.$$

It should be noted that Margolin and Light's selection of Kullback's "independence" test statistic (2I) was not appropriate for the homogeneity test [e.g., see Gokhale and Kullback (1978b)].

2.3.11 Camilli and Hopkins (1978)

Camilli and Hopkins presented a Monte Carlo study of the adequacy of the Pearson X^2 test, with and without Yates' (1934) correction for continuity, for 2×2 contingency tables. They considered the three common sampling models: (1) both marginals fixed, (2) one marginal fixed, and (3) no marginals fixed. Model (1) corresponds to Fisher's exact test probabilities based on the hypergeometric distribution. Camilli and Hopkins stated that this distribution was seldom seen in actual research studies and concluded that there was a serious problem in the use of Fisher's exact test since the data usually had not resulted from fixed marginals. Model (2) was the model used by Roscoe and Byars

(1971) and model (3) by Craddock and Flood (1970). Camilli and Hopkins generated 10,000 random tables using specified probability structures under models (2) and (3) and estimated exact significance levels for the Pearson X^2 , with and without Yates' correction, at the nominal .01, .05, and .10 levels.

Camilli and Hopkins concluded that the tests without Yates' correction, for homogeneity [model (2)] and independence [model (3)], gave accurate, though somewhat conservative, type I error probability statements when the sample size was twenty or more, even when the expected cell values in one or two cells were as low as one or two. Yates' correction was shown to be totally inadequate under models (2) and (3), causing the statistic to be ultra-conservative with a corresponding loss of power. This last conclusion agrees with those of other studies [e.g., Conover (1974)].

2.3.12 Larntz (1978)

Larntz considered a variety of categorical data models with and without estimated parameters and used exact and Monte Carlo methods to determine exact levels of significance for three "goodness-of-fit" statistics: likelihood ratio G^2 , Pearson X^2 , and a variant of the Freeman-Tukey T^2 . He tested five models:

- (1) a multinomial model
- (2) a group helping model in a 3×2 table with fixed column margins
- (3) A no-association model with both margins fixed (hypergeometric) in a 3×5 table

- (4) a quasi-independence model in a 5×5 table
- (5) a no second-order interaction model in a $3 \times 3 \times 3$ table.

Larntz's conclusions were that the approximate G^2 and T^2 tests yielded exact levels that were typically in excess of nominal levels with moderate expected values (1.5-4.0) and that the Pearson X^2 statistic closely attained exact levels for a wide range of sample sizes, expected values, and models. These results with regard to contingency tables were consistent with those of Odoroff (1970) and Margolin and Light (1974) for G^2 and X^2 .

Larntz provided an explanation for the superiority of X^2 in small samples. He calculated the minimum contribution to each statistic for observed cell counts of zero and one with varying cell expectations. The Pearson X^2 was least affected by the small observed counts.

2.3.13 Miller (1979)

Miller also investigated the Pearson X^2 and the likelihood ratio G^2 statistics when expected values were small and parameters were estimated from the data. In particular, he considered tests of independence in two-way contingency tables (2×2 , 5×2 , and 5×5) when one or more of the expected values were five or less. He evaluated the X^2 and G^2 statistics, with and without Yates continuity correction, when used to test both independence and homogeneity. Monte Carlo methods were used to form the empirical distributions of these statistics and compare them with the chi-squared distribution at nominal levels .01, .05, and .10.

Miller's results indicated that the Pearson X^2 statistic was robust with respect to small expected values. It performed well when expected values were as low as two and even as low as 0.5 when the

number of such cells was small with respect to the total number of cells. The likelihood ratio G^2 statistic usually performed well for minimum expected values of five, but its performance was inconsistent for values less than five. In general, Yates' continuity correction gave a more conservative test. These results agreed with those of Margolin and Light (1974), Camilli and Hopkins (1978), and Larntz (1978). In the 2×5 and 5×5 tables, Miller noted a slightly poorer performance of the Pearson X^2 at the equiprobable vector when expected values were very small. He concluded that this was because all the expected values were small with no larger values for balance. Actually, his criteria required that sample sizes for the uniform vectors be smaller, and this alone would probably cause a poorer performance.

2.3.14 Wang (1979)

Wang extended Odoroff's study to consider tests for main effects when zero interaction was assumed in the manner of "partitioning of conditional independence" described by Goodman (1970). As in Odoroff's study, the results are limited to $r \times s \times 2$ tables under the product-binomial sampling scheme. Wang used eight of the twelve Odoroff tests, excluding those associated with likelihood ratio which performed very poorly for Odoroff. Wang used Odoroff's Monte Carlo procedures and the same $2 \times 2 \times 2$ and $3 \times 2 \times 2$ tables. The eight tests were evaluated with respect to the exact conditional levels of significance (at nominal .05 and .01 levels) in rejecting the null hypothesis of zero column (row) effect, assuming zero row by column effect at specified nominal levels of rejection (.10, .25, .50, and .75).

From among the eight tests, the minimum logit statistic with one of the minimum logit estimation techniques and the Pearson X^2 with another of the minimum logit estimation techniques were identified as best with respect to closeness of approximation and simplicity of computational procedure. The first test was the same best test found by Odoroff in testing for independence. Wang used the same minimum cell expectation (MCE) computations as Odoroff in reporting his results.

A different aspect of Wang's study was the inclusion of varied levels for rejection of the independence hypothesis. In general, as the MCE increased, lower levels for rejection of the independence hypothesis were allowed.

2.3.15 Cox and Plackett (1980)

Cox and Plackett suggested three ways for improving current asymptotic methods for analyzing contingency tables. According to Cox and Plackett, these methodologies provide: (i) better approximations to moments of cell frequencies (expected values) than those derived from the iterative scaling procedure, (ii) exact distributions obtained by enumerating all tables consistent with given marginal totals, and (iii) simulated random sampling for tables where exact enumeration is impossible". Their study was restricted to the consideration of tables with fixed marginals using "exact" distributions as a criterion for comparison.

To compare their approximations in (i), Cox and Plackett used a 2×2 and 2×3 table with arbitrary cross-product ratios and a 2^k ($k=3$) table with arbitrary k -way interaction. Their approximations were clearly superior to the asymptotic maximum likelihood; however, these

new approximations had little affect on the asymptotic methods of inference.

Using several examples ($3 \times 3 \times 3$, $2 \times 2 \times 18$, 2×15 , $2 \times 2 \times t$), Cox and Plackett demonstrated that complete enumeration or Monte Carlo methods could be used in a variety of contingency table cases. They concluded that exact and Monte Carlo methods could be used with small samples, and then "avoid the mass of complicated recommendations...".

In their examples Cox and Plackett compared the "exact" distributions of the Pearson X^2 and likelihood ratio G^2 with the chi-squared distribution. In general, their study agreed with those of others, that the Pearson X^2 was closer to the chi-squared distribution than the likelihood-ratio G^2 .

2.3.16 Discussion of Limitations

Table 3 provides a summary of the previous studies reviewed. This subsection discusses the limitations of these studies with respect to table designs, hypotheses, statistics investigated, and assessment criteria.

Many of these studies, eight out of 15, restricted their evaluation to the Pearson X^2 statistic. Six others included the log-likelihood ratio G^2 . One used Kullback's minimum discrimination information statistic, 2I, but misapplied it to the homogeneity test. Some of the studies included other statistics, but the use of these statistics often was restricted to small tables and/or certain types of hypotheses, such as Goodman and Kruskal's L and Freeman and Tukey's T^2 . None of the studies included the GSK procedure and statistic in their evaluation.

The studies were very selective with respect to the hypotheses

they considered. Some considered only the homogeneity hypothesis. Others considered the independence hypothesis but required all the marginals to be fixed, which severely limited the application of their conclusions. The studies that did consider the general independence hypothesis under multinomial sampling were usually very restrictive in some other aspect. Only three studies included the no second-order interaction hypothesis, and none evaluated independence hypotheses for more than two variables.

While most of the studies used fairly reasonable criteria for assessment, some of the studies based their comparisons on Fisher's exact test or similar exact tests. Fisher's exact test is known to be conservative and is based on fixed marginals. As has been suggested [e.g., Berkson (1978)], it may not provide a reasonable criterion for comparison.

The studies were all somewhat restrictive in the size of the tables (number of cells), number of dimensions, and sample size considered. This is to be expected because of the nature of the problem and computer time limitations. Many of the early tests considered only one or two table sizes and few sample sizes. One of the larger studies (with respect to number of different tables), Roscoe and Byars (1971), only evaluated the homogeneity hypothesis. Only three of the studies included three-way tables. Few of the studies considered more than four or five different sample sizes.

The most significant limitation of these studies concerned the underlying probability structures. A few of the studies were restricted to the equiprobable model. Most of the others included only two or

three different probability structures. The homogeneity tests generally had tables which restricted the fixed marginals to be equal or symmetric. Some of the studies used single examples with only one fixed probability structure.

None of the studies combined an extensive number of different table sizes, sample sizes, and probability vectors. With respect to the homogeneity hypothesis, Roscoe and Byars (1971) probably did the best, combining 10 types of two-dimensional tables, six sample sizes, and three probability structures. Odoroff (1970) considered a reasonable range of table designs with respect to his MCE calculations, but his study was limited to the "no second-order" interaction hypothesis. None of the independence studies provided a reasonable range of table designs.

Especially important in this thesis are comparisons across tables under the independence hypothesis. With the exception of Craddock and Flood (1970), no study provided a means for these comparisons. Craddock and Flood's study was limited to the equiprobable model. Also, their conclusion for the performance of the Pearson X^2 in larger tables was based on a very subjective analysis. What appears to be needed is a more objective criterion for comparisons across tables and a more extensive consideration of table sizes, sample sizes, and, especially, probability vectors. This will be provided by the research of this thesis.

Table 3. Summary of Previous Studies

Study	Table Designs	Hypotheses	Statistics Investigated	Assessment Criteria	Conclusions
•Levontin and Felsenstein (1965)	•2 x 5 (N=20, 40, 60) •2 x 10 (N=40, 80, 120) •both sets of margins fixed •symmetric column margins	•homogeneity	•Pearson χ^2	•Monte Carlo (5000) •Estimated exact distribution χ^2 compared to χ^2 (.10, .05, .02, .01)	•Pearson χ^2 conservative for 5 or more d.f. •Robust for expected values as low as 1
•Maynam and Leone (1965)	•2 x 3 •3 x 3 •N=10, 15 •no margins fixed •one set fixed •both sets fixed •equiprobable	•homogeneity •independence	•Pearson χ^2	•Exact distribution points calculated based on sampling model •compared to χ^2 cumulative χ^2	•Pearson χ^2 well-approximated by χ^2 up to cumulative probability of 0.9 • χ^2 overestimates "rather badly" exact probabilities in tail area •results similar for all 3 sampling models
•Craddock (1966)	•3 x 3 •N=10, 15, 20, 25, 30, 40, 50, 100 •equiprobable •no margins fixed	•independence	•Pearson χ^2	•Monte Carlo (10,000) •Histogram of χ^2 statistic •percentile values compared to χ^2	•Excellent agreement of percentile values down to N=30 •Can use Pearson χ^2 as test for N as low as 10
•Sugiura and Orake (1968)	•Examples 2 x 2 Rao (1952) 2 x 2 Symmetric 2 x 2 large marginals 2 x 3 Yates (1934)	•independence	•12 Statistics (see Section 2.3.4)	•Fishers' Exact test for 2 x 2 •Exact tests based on χ^2 and LR ordering for 2 x 3	•(5), (10), (11), (12) somewhat better for 2 x 2 tables though more conservative than exact levels •Cart's methods (11) & (12) were conservative for 2 x 3 table •LR modification (6) always better than LR(4) •Nass' methods (8) & (9) not good •Other methods similar to χ^2

Table 3. Continued

Study	Table Designs	Hypotheses	Statistics Investigated	Assessment Criteria	Conclusions
•Odoroff (1970)	<ul style="list-style-type: none"> •$2 \times 2 \times 2$ •$3 \times 2 \times 2$ •one set of two-way margins fixed •specified Minimum Cell Expectations (MCE) 	<ul style="list-style-type: none"> •"no second-order" interaction 	<ul style="list-style-type: none"> •3 Statistics (i) <ul style="list-style-type: none"> 1. min logit chi-square 2. Pearson χ^2 3. LR •4 Estimates (j) to χ^2 at .05 level 1. max. likelihood 2-4. min. logit (j) •12 Tests (χ^2_{1j}) 	<ul style="list-style-type: none"> •Exact - $2 \times 2 \times 2$ •Monte Carlo - $3 \times 2 \times 2$ (2000) •Exact distributions of tests compared to χ^2 at .05 level •(0.04, 0.06) good •reported with respect to MCE 	<ul style="list-style-type: none"> •Small sample properties $\chi^2_{13}, \chi^2_{23}, \chi^2_{24}$ best •Ease of computation χ^2_{13} best •χ^2_{23}, χ^2_{24} good •No power differences
•Craddock and Flood (1970)	<ul style="list-style-type: none"> •$3 \times 2 \rightarrow 5 \times 5$ •equiprobable •M=12, 50 \rightarrow M=16, 100 	<ul style="list-style-type: none"> •independence 	<ul style="list-style-type: none"> •Pearson χ^2 	<ul style="list-style-type: none"> •Monte Carlo (10,000) •Histograms of χ^2 •Percentiles compared to χ^2 	<ul style="list-style-type: none"> •Five "Rule of Thumb" too conservative •behavior of χ^2 better for larger tables
•March (1970)	<ul style="list-style-type: none"> •2×3 •M=6(1) 30, 36, 42 •all possible tables with expected values > 1 •Fixed margins 	<ul style="list-style-type: none"> •independence 	<ul style="list-style-type: none"> •Pearson χ^2 	<ul style="list-style-type: none"> •Enumeration to calculate cumulative probs. and compare to exact hypergeometric •underestimates/overestimates 	<ul style="list-style-type: none"> •Statistic improves as N increases •χ^2 poor approximation to general hypergeometric •χ^2 good as test of significance in nominal regions even with low expected values
•Yarnold (1970)	<ul style="list-style-type: none"> •multinomial 	<ul style="list-style-type: none"> •goodness-of-fit 	<ul style="list-style-type: none"> •Pearson χ^2 	<ul style="list-style-type: none"> •Rule of thumb 5r/s s = No. classes r = No. expectations < 5 	<ul style="list-style-type: none"> •Rule of thumb 5r/s s = No. classes r = No. expectations < 5
•Roscoe and Byars (1971)	<ul style="list-style-type: none"> •$2 \times 2 \rightarrow 5 \times 5$ •M=10, 15, 20, 30, 50, 100 •fixed and equal row marginals •uniform •moderate skewness •extreme skewness 	<ul style="list-style-type: none"> •homogeneity 	<ul style="list-style-type: none"> •Pearson χ^2 	<ul style="list-style-type: none"> •Monte Carlo (10,000) •estimated exact levels of significance compared to nominal (.05, .01) •Cochran's (1952) criteria 	<ul style="list-style-type: none"> •"average expected values" •uniform - ≥ 2 at .05 ≥ 4 at .01 •moderate - ≥ 4 at .05 ≥ 6 at .01 •extreme - ≥ 6 at .05 ≥ 10 at .01

Table 3. Continued

Study	Table Designs	Hypotheses	Statistics Investigated	Assessment Criteria	Conclusions
Margolin and Light (1974)	<ul style="list-style-type: none"> • 3×2 • $x_{.1} = x_{.2} = N/2$ fixed • 11 probability vectors for p_1. 	<ul style="list-style-type: none"> • homogeneity 	<ul style="list-style-type: none"> • Pearson χ^2 • Kulback 21 • Goodman & Kruskal L • Margolin & Light C 	<ul style="list-style-type: none"> • Exact tail areas (levels of significance) calculated for .10, .05, .025, .01 • χ^2 conditional and unconditional tests based on zero marginals 	<ul style="list-style-type: none"> • L poor test (liberal) • C better than χ^2 • C, χ^2 better than 21 • C, χ^2 better near equiprobable • χ^2 generally conservative • C, χ^2, 21 better conditional tests • (2 fn 2) $\chi^2 > 21 > \chi^2$
Camilli and Hopkins (1978)	<ul style="list-style-type: none"> • 2×2 • N=20, 50, 100 • (1) fixed margins • (2) one margin fixed • $x_1 = 5, 10$ • (3) none fixed • $p_1 = .500, .707, .794$ • N=20 	<ul style="list-style-type: none"> • homogeneity (model (2)) • independence (model (3)) 	<ul style="list-style-type: none"> • Pearson χ^2 • Yates' (1934) continuity correction 	<ul style="list-style-type: none"> • Monte Carlo (10,000) • estimated exact levels of significance at nominal .01, .05, .10 	<ul style="list-style-type: none"> • Yates correction poor, ultra-conservative • Pearson χ^2 good w/small expectations (1 or 2) for $N > 20$, somewhat conservative
Larntz (1978)	<ul style="list-style-type: none"> • (1) multinomial • (2) 3×2 one set margins fixed • (3) 3×5 fixed margins • (4) 5×5 incomplete • (5) $3 \times 3 \times 3$ 	<ul style="list-style-type: none"> • (1) goodness-of-fit • (2) homogeneity • (3) independence • (4) quasi-independence • (5) no second-order interaction 	<ul style="list-style-type: none"> • Pearson χ^2 • Likelihood Ratio G^2 • Freeman-Tukey T^2 	<ul style="list-style-type: none"> • exact & Monte Carlo (1000, 2000, 10000) • exact levels of significance at .05 nominal level • good .05 \pm .02 • MCE 	<ul style="list-style-type: none"> • G^2, T^2 liberal for expected values 1.5 - 4.0 • χ^2 good exact levels across models • χ^2 less affected by small observed values
Miller (1979)	<ul style="list-style-type: none"> • 2×2 • 2×5 • 5×5 • various marginal probabilities • MCE (.5, 2, 5) 	<ul style="list-style-type: none"> • independence 	<ul style="list-style-type: none"> • Pearson χ^2 • Likelihood Ratio G^2 • Yates' continuity correction 	<ul style="list-style-type: none"> • Monte Carlo (2000) • .95 percent C.I. criteria • estimated exact levels compared to nominal .01, .05, .10 	<ul style="list-style-type: none"> • Pearson χ^2 good for expected values as low as 2 • G^2 good for expected values as low as 5 • Yates correction more conservative

Table 3. Continued

Study	Table Designs	Hypotheses	Statistics Investigated	Assessment Criteria	Conclusions
Wang (1979)	see Odoroff (1970)	main effects given zero interaction	8 tests of Odoroff (not LR)	<ul style="list-style-type: none"> Monte Carlo (2000) estimated exact conditional levels compared to nominal (.01, .05) .05 + .01 good .01 + .005 	<ul style="list-style-type: none"> χ^2_{22} and χ^2_{13} best χ^2_{13} agreed with Odoroff as MCE increases lower levels of independence rejection allowed
Cox and Plackett (1980)	<ul style="list-style-type: none"> $2 \times 2, 2^3, 2 \times 3$ some marginals fixed $.3^3, 2 \times 2 \times 18, 2 \times 15, 2 \times 2 \times t$ 	<ul style="list-style-type: none"> homogeneity independence 	<ul style="list-style-type: none"> Pearson χ^2 Likelihood Ratio G^2 	<ul style="list-style-type: none"> comparisons made to exact levels of significance exact & Monte Carlo (1000, 10000) 	<ul style="list-style-type: none"> Pearson χ^2 better new approximations given for expected values use exact and Monte Carlo methods for inference testing

CHAPTER III

CURRENT METHODOLOGIES

This chapter presents and discusses the most popular methodologies for the analysis of contingency tables. The first four sections describe and relate the underlying sampling models, the log-linear models, the hypothesis tests, and the maximum likelihood estimates. Particular emphasis is given to two- and three-way tables and hypotheses of independence, homogeneity, no second-order interaction, and complete independence. The last three sections present the three competing statistics: the Pearson X^2 , the Kullback minimum discrimination information statistic, and the Grizzle, Starmer, and Kock (GSK) statistic. The theoretical developments of the Kullback and GSK statistics are presented in detail.

3.1 Sampling Models

Most contingency tables are a result of sampling from a larger population. The type of sampling model will determine a sampling distribution for the observations. The four most common sampling distributions are the Poisson, the multinomial, the product-multinomial, and the hypergeometric. These sampling models and corresponding distributions relate directly to the restrictions placed on the total sample size and marginal sums.

If there are no restrictions placed on the total sample size or marginal sums, and observations are made over a fixed period of time,

each cell has an independent Poisson distribution. The density function for each cell of the two-way $r \times s$ table is

$$f(x_{ij}) = \prod_{i=1}^r \prod_{j=1}^s (m_{ij}^{x_{ij}} e^{-m_{ij}}) / x_{ij}! ;$$

$$i=1,2,\dots,r; j=1,2,\dots,s. \quad (3-1)$$

If only the total sample size (N) is fixed, the distribution of the cells is multinomial, and the density function for each cell of an $r \times s$ table is

$$f(x_{ij}) = N! N^{-N} \prod_{i=1}^r \prod_{j=1}^s m_{ij}^{x_{ij}} / x_{ij}! ;$$

$$i=1,2,\dots,r; j=1,2,\dots,s. \quad (3-2)$$

If each category of one of the variables has a fixed sample size (marginal sum), then the sampling distribution is product-multinomial, and the density function for each cell of an $r \times s$ table is

$$f(x_{ij} | x_{.j}) = \prod_{j=1}^s [(x_{.j}! / \prod_{i=1}^r x_{ij}!) \prod_{i=1}^r (m_{ij} / x_{.j})^{x_{ij}}];$$

$$i=1,2,\dots,r; j=1,2,\dots,s. \quad (3-3)$$

If each category of all the variables have a fixed sample size (i.e., all marginal sums are fixed), then the sampling distribution is hypergeometric, and the density function for each cell of an $r \times s$ table is

$$f(x_{ij} | x_{i.}, x_{.j}) = \frac{\prod_{i=1}^r x_{i.}! \prod_{j=1}^s x_{.j}!}{N! \prod_{i=1}^r \prod_{j=1}^s x_{ij}!};$$

$$i=1,2,\dots,r; j=1,2,\dots,s. \quad (3-4)$$

As might be expected, these distributions are highly interrelated. If X_{ij} are independent Poisson random variables with density function given by (3-1), then the conditional distributions of the X_{ij} given $N = \sum_{i=1}^r \sum_{j=1}^s X_{ij}$ are the multinomial distribution of (3-2). When sampling under the product-multinomial distributions of (3-3), the distributions of the marginals $(x_{.j})$ are multinomial with density functions similar to (3-2),

$$f(x_{.j}) = N! N^{-N} \prod_{j=1}^s m_{ij}^{x_{ij}} / x_{.j}!;$$

$$j=1,2,\dots,s. \quad (3-5)$$

When sampling under the product-multinomial distribution with $x_{.j}$ fixed, and then fixing the other marginals $(x_{i.})$, the corresponding conditional distributions are hypergeometric as given in (3-4).

The multinomial sampling model will be used for the study in this thesis. As will be discussed in Section 3, this model is appropriate for the independence hypotheses of this study, while the product-multinomial model corresponds to homogeneity hypotheses. Little is to be gained by separately considering Poisson or hypergeometric sampling. Under Poisson sampling, the hypotheses, the maximum likelihood estimates, and the chi-square statistics correspond exactly to either the product-multinomial or multinomial cases. As previously noted in Chapter II, the hypergeometric sampling plan is rarely used in practice. Under those rare situations where both marginals of a two-way table are fixed, Fisher's "exact" test, based on the hypergeometric probabilities, is appropriate for the independence hypothesis.

3.2 Log-Linear Model

As previously mentioned in Chapter II, the log-linear model forms the basis for much of the work in contingency table analysis. Maximum likelihood estimates and minimum discrimination information estimates, under the sampling models discussed and for appropriate hypotheses, have been fully developed for this model and are relatively easy to obtain; whereas, other models have suffered from the complexity of solving for appropriate estimates.

The GSK procedure does allow for both the linear and log-linear models to be handled in a similar manner. G. C. Koch and others have extensively used the linear (additive) model [e.g., Johnson and Koch (1970), Koch and Reinfurt (1971), and Koch, Johnson, and Tolley (1972)]. The advantage of this model seems to be associated with incomplete

tables, where structured zero cells are present [e.g., see Koch and Reinfurt (1971), Bishop, Fienberg, and Holland (1975, Chapter 5), and Chen and Fienberg (1976)]. In a recent book Forthofer and Lehnen (1981) state that the log-linear (multiplicative) model is preferred when events are "extreme"; i.e., when some cell probabilities are relatively small or large (less than 0.2 or greater than 0.8). When dealing with large contingency tables, frequently, all the cell probabilities are small, especially under multinomial sampling.

The log-linear model with its associated multiplicative interactions will be used for this study. The model is directly analogous to the linear model for analysis of variance (ANOVA). For a two-way $r \times s$ contingency table the complete or fully saturated log-linear model is

$$\ln p_{ij} = u + u_{1(i)} + u_{2(j)} + u_{12(ij)};$$

$$i=1,2,\dots,r; j=1,2,\dots,s; \quad (3-6)$$

where $u \equiv$ the grand mean of the logarithms of the probabilities; i.e.,

$$u = (1/rs)(\ln p_{11} + \ln p_{12} + \dots + \ln p_{rs}). \quad (3-7)$$

Also, $u + u_{1(i)} \equiv$ the mean of the logarithms of the probabilities at level i for the first variable; i.e.,

$$u + u_{1(i)} = (1/s)(\ln p_{i1} + \ln p_{i2} + \dots + \ln p_{is});$$

$$i=1,2,\dots,r. \quad (3-8)$$

Similarly, $u + u_{2(j)} \equiv$ the mean of the logarithms of the probabilities at level j for the second variable; i.e.,

$$u + u_{2(j)} = (1/r)(\ln p_{1j} + \ln p_{2j} + \dots + \ln p_{rj});$$

$$j=1,2,\dots,s. \quad (3-9)$$

As defined, $u_{1(i)}$ and $u_{2(j)}$ represent deviations from the grand mean. Summing (3-8) over i ,

$$\sum_{i=1}^r (u + u_{1(i)}) = \frac{1}{s} \sum_{i=1}^r (\ln p_{i1} + \ln p_{i2} + \dots + \ln p_{is})$$

$$ru + \sum_{i=1}^r u_{1(i)} = \frac{1}{s} (\ln p_{11} + \ln p_{12} + \dots + \ln p_{rs})$$

$$u + \frac{1}{r} \sum_{i=1}^r u_{1(i)} = u$$

$$\sum_{i=1}^r u_{1(i)} = 0.$$

Similarly, summing (3-9) over j , $\sum_{j=1}^s u_{2(j)} = 0$. These results are analogous to the familiar constraints for the ANOVA linear statistical model. In a similar manner the constraints, $\sum_i u_{12(ij)} = \sum_j u_{12(ij)} = 0$,

are placed on the $u_{12(ij)}$, and these terms represent the interaction between the variables, analogous to interaction in ANOVA. Continuing the ANOVA analogy and solving for the terms of the model, from (3-7), (3-8), and (3-9) the main effects are

$$u_{1(i)} = \frac{1}{s} \sum_{j=1}^s \ln p_{ij} - \frac{1}{rs} \sum_{i=1}^r \sum_{j=1}^s \ln p_{ij} ;$$

$i=1,2,\dots,r$ (3-10)

and

$$u_{2(j)} = \frac{1}{r} \sum_{i=1}^r \ln p_{ij} - \frac{1}{rs} \sum_{i=1}^r \sum_{j=1}^s \ln p_{ij} ;$$

$j=1,2,\dots,s$ (3-11)

Using (3-6), (3-7), (3-10), and (3-11),

$$\begin{aligned} u_{12(ij)} &= \ln p_{ij} - u - u_{1(i)} - u_{2(j)} \\ &= \ln p_{ij} - u - \frac{1}{s} \sum_{j=1}^s \ln p_{ij} + u - \frac{1}{r} \sum_{i=1}^r \ln p_{ij} + u \\ &= \ln p_{ij} - \frac{1}{s} \sum_{j=1}^s \ln p_{ij} - \frac{1}{r} \sum_{i=1}^r \ln p_{ij} \\ &\quad + \frac{1}{rs} \sum_{i=1}^r \sum_{j=1}^s \ln p_{ij}. \end{aligned} \tag{3-12}$$

Degrees of freedom for these terms are calculated exactly as in ANOVA. There are a total of $r \cdot s$ degrees of freedom, one associated with each observation or cell. Each u term has one degree of freedom associated with each independent parameter. The constraints reduce the number of independent parameters. Table 4 lists the degrees of freedom for the $r \times s$ table.

Table 4. Degrees of Freedom: $r \times s$ Table

<u>u-term</u>	<u>degrees of freedom</u>
u	1
u_1	$r-1$
u_2	$s-1$
u_{12}	$(r-1)(s-1)$
Total	rs

The extension to higher-way tables follows as easily as the extension of the linear model for ANOVA. The only problems involved are the interpretations of the higher-way interaction terms. An excellent discussion is presented in Bishop, Fienberg, and Holland (1975, Section 2.4).

Frequently, it is more convenient to work in the frequency spectrum (x_{ij} and m_{ij}). When this is the case, the log-linear model can easily be modified. For multinomial sampling

$$m_{ij} = Np_{ij}.$$

Taking logarithms of both sides,

$$\ln m_{ij} = \ln N + \ln p_{ij}.$$

The log-linear model then becomes

$$\begin{aligned} \ln m_{ij} &= u + u_{1(i)} + u_{2(j)} + u_{12(ij)} + \ln N \\ &= u' + u_{1(i)} + u_{2(j)} + u_{12(ij)}, \end{aligned} \quad (3-13)$$

where

$$u' = u + \ln N.$$

For product-multinomial sampling

$$m_{ij} = m_{i.} p_{ij}$$

and

$$\begin{aligned} \ln m_{ij} &= \ln m_{i.} + \ln p_{ij} \\ &= u^{(i)} + u_{1(i)} + u_{2(j)} + u_{12(ij)}, \end{aligned}$$

where

$$u^{(i)} = u + \ln m_{i.}.$$

3.3 Hypothesis Tests

The ultimate goal in the analysis of contingency tables is to determine the relationships among the variables. The approaches taken are problem dependent in that they depend on what particular relationship is of concern. Sometimes a simple procedure of "smoothing" the cells of a contingency table to correspond to some fixed constraints or to some other set of data is enough to indicate the desired relationships. Other times certain estimates of the cells are calculated to show patterns or trends. More often than not a hypothesis testing procedure is necessary to explore variable relationships. The types of hypotheses are numerous, and the literature is filled with extensive discussions and interpretations. This study will be limited to examining the small sample properties of current methodologies as they relate to hypotheses of independence, or "no-interaction", under multinomial sampling. These hypotheses will be discussed in this section together with the contrasting homogeneity hypotheses under product-multinomial sampling.

All these hypotheses correspond to setting certain terms of the log-linear model equal to zero. To demonstrate these relationships, consider the 2×2 table of probabilities in Figure 3. Under multinomial sampling $\sum_i \sum_j p_{ij} = 1$, and the hypothesis of concern is independence, or no interaction, between the variables A and B. The marginal probabilities $p_{1.}$, $p_{2.}$, $p_{.1}$, and $p_{.2}$ are the unconstrained probabilities of being in a certain category of a variable (e.g., $P(A=1) = p_{1.}$). The classic sense of independence would require that $P(AB) = P(A)P(B)$. The corresponding hypothesis statement for the 2×2 contingency

		A		
		1	2	
A	1	p_{11}	p_{12}	$p_{1.}$
	2	p_{21}	p_{22}	$p_{2.}$
		$p_{.1}$	$p_{.2}$	

Figure 3. 2×2 Table of Probabilities

table is

$$H_0: p_{ij} = p_{i.}p_{.j}; i=1,2; j=1,2. \quad (3-14)$$

For the 2×2 table it is easily seen that the hypothesis,

$$H_0: p_{11} = p_{1.}p_{.1}, \text{ is sufficient for (3-14):}$$

$$\begin{aligned} p_{11} &= p_{1.}p_{.1} \\ &= p_{1.}(1-p_{.2}) = p_{1.} - p_{1.}p_{.2}, \end{aligned}$$

but $p_{11} = p_{1.} - p_{12}$; therefore,

$$p_{12} = p_{1.}p_{.2}.$$

$$\begin{aligned} p_{11} &= p_{1.}p_{.1} \\ &= (1-p_{2.})p_{.1} = p_{.1} - p_{2.}p_{.1}, \end{aligned}$$

but $p_{11} = p_{.1} - p_{21}$; therefore,

$$p_{21} = p_{2.}p_{.1}.$$

$$\begin{aligned}
 p_{11} &= p_{1.} p_{.1} = (1-p_{2.})(1-p_{.2}) \\
 &= 1 - p_{2.} - p_{.2} + p_{2.} p_{.2},
 \end{aligned}$$

but

$$\begin{aligned}
 p_{11} &= 1 - p_{12} - p_{21} - p_{22} \\
 &= 1 - (p_{.2} - p_{22}) - (p_{2.} - p_{22}) - p_{22} \\
 &= 1 - p_{2.} - p_{.2} + p_{22};
 \end{aligned}$$

therefore,

$$p_{22} = p_{2.} p_{.2}.$$

From (3-6) the log-linear model for the 2×2 table is

$$\ln p_{ij} = u + u_{1(i)} + u_{2(j)} + u_{12(ij)}; \quad i=1,2; \quad j=1,2$$

with constraints,

$$u_{1(1)} + u_{1(2)} = 0$$

$$u_{2(1)} + u_{2(2)} = 0$$

$$u_{12(11)} + u_{12(21)} = 0$$

$$u_{12(12)} + u_{12(22)} = 0$$

$$u_{12(11)} + u_{12(12)} = 0$$

$$u_{12(21)} + u_{12(22)} = 0.$$

The following relations exist:

$$u_{1(1)} = -u_{1(2)}, \quad u_{2(1)} = -u_{2(2)},$$

and

$$u_{12(11)} = -u_{12(12)} = -u_{12(21)} = u_{12(22)}.$$

Reparameterizing by letting

$$u_{1(1)} = u_1, \quad u_{2(1)} = u_2, \quad \text{and} \quad u_{12(11)} = u_{12},$$

the logarithms of the four cell probabilities are

$$\begin{aligned} \ln p_{11} &= u + u_1 + u_2 + u_{12} \\ \ln p_{12} &= u + u_1 - u_2 - u_{12} \\ \ln p_{21} &= u - u_1 + u_2 - u_{12} \\ \ln p_{22} &= u - u_1 - u_2 + u_{12}. \end{aligned} \tag{3-15}$$

Solutions for the "u" parameters can be obtained from (3-15). Summing the four equations,

$$4u = \ln p_{11} + \ln p_{12} + \ln p_{21} + \ln p_{22};$$

so that,

$$u = \frac{1}{4} (\ln p_{11} + \ln p_{12} + \ln p_{21} + \ln p_{22}). \tag{3-16}$$

Substituting (3-16) into the first and second, and summing,

$$2u_1 + 2\left[\frac{1}{4}(\ln p_{11} + \ln p_{12} + \ln p_{21} + \ln p_{22})\right] = \ln p_{11} + \ln p_{12} ;$$

therefore,

$$u_1 = \frac{1}{4}(\ln p_{11} + \ln p_{12} - \ln p_{21} - \ln p_{22}). \quad (3-17)$$

Substituting (3-16) into the first and third, and summing,

$$2u_2 + 2\left[\frac{1}{4}(\ln p_{11} + \ln p_{12} + \ln p_{21} + \ln p_{22})\right] = \ln p_{11} + \ln p_{21} ;$$

therefore,

$$u_2 = \frac{1}{4}(\ln p_{11} - \ln p_{12} + \ln p_{21} - \ln p_{22}). \quad (3-18)$$

Summing the first and fourth equations, and subtracting the second and third equations,

$$4u_{12} = \ln p_{11} - \ln p_{12} - \ln p_{21} + \ln p_{22} ;$$

so that,

$$u_{12} = \frac{1}{4}(\ln p_{11} - \ln p_{12} - \ln p_{21} + \ln p_{22}). \quad (3-19)$$

Under the hypothesis of no-interaction, (3-14),

$$\begin{aligned}
 u_{12} &= \frac{1}{4} (\ln p_{1.} p_{.1} - \ln p_{1.} p_{.2} - \ln p_{2.} p_{.1} + \ln p_{2.} p_{.2}) \\
 &= \frac{1}{4} (\ln p_{1.} + \ln p_{.1} - \ln p_{1.} - \ln p_{.2} - \ln p_{2.} - \ln p_{.1} \\
 &\quad + \ln p_{2.} + \ln p_{.2}) \\
 &= 0.
 \end{aligned}$$

Conversely, if $u_{12} = 0$ then

$$\ln p_{11} - \ln p_{12} - \ln p_{21} + \ln p_{22} = 0$$

and

$$\ln \left(\frac{p_{11} p_{22}}{p_{12} p_{21}} \right) = 0,$$

which implies that

$$\frac{p_{11} p_{22}}{p_{12} p_{21}} = 1,$$

or

$$p_{11}p_{22} = p_{12}p_{21}.$$

Making substitutions,

$$p_{11}(1-p_{11}-p_{12}-p_{21}) = (p_{1.}-p_{11})(p_{.1}-p_{11})$$

$$p_{11}(1-p_{11}-(p_{1.}-p_{11}) - (p_{.1}-p_{11})) = p_{1.}p_{.1} - p_{11}(p_{1.}+p_{.1}-p_{11})$$

$$p_{11}(1-p_{1.}-p_{.1}+p_{11}) = p_{1.}p_{.1} - p_{11}(p_{1.}+p_{.1}-p_{11})$$

$$p_{11} = p_{1.}p_{.1},$$

and independence, (3-14), follows. Thus,

$$H_0: u_{12} = 0 \quad (3-20)$$

is equivalent to (3-14) for the 2×2 case. In general, for the $r \times s$ case

$$H_0: u_{12}(ij) = 0; i=1,2,\dots,r; j=1,2,\dots,s. \quad (3-21)$$

Consider now the 2×2 table under product-multinomial (product-binomial) sampling, where the marginal observations for one of the variables are fixed, say $x_{1.}$ and $x_{2.}$. Thus, the corresponding probabilities, $p_{1.}$ and $p_{2.}$, are equal to one, and the 2×2 table is given in Figure 4.

p_{11}	$1-p_{11}$	1
p_{21}	$1-p_{21}$	1

Figure 4. 2×2 Table: Product-Binomial Sampling

The hypothesis of concern is homogeneity; that is,

$$H_0: p_{11} = p_{21} = p \text{ or } p_{12} = p_{22} = 1 - p, \quad (3-22)$$

where p is usually not specified. The complete log-linear model, (3-6), under product-multinomial sampling is

$$\ln p_{ij} = u + u_{1(i)} + u_{2(j)} + u_{12(ij)}; \quad i=1,2; \quad j=1,2;$$

where now $p_{12} = 1 - p_{11}$ and $p_{22} = 1 - p_{21}$. Using the parameterization in (3-15) and the relations in (3-7), (3-8), and (3-9),

$$\begin{aligned} u &= \frac{1}{4} (\ln p_{11} + \ln (1-p_{11}) + \ln p_{21} + \ln (1-p_{21})) \\ &= \frac{1}{4} (\ln p_{11}(1-p_{11}) + \ln p_{21}(1-p_{21})); \end{aligned} \quad (3-23)$$

$$u + u_1 = \frac{1}{2} (\ln p_{11} + \ln p_{12}) = \frac{1}{2} \ln p_{11}(1-p_{11});$$

so that,

$$u_1 = \frac{1}{4} (\ln p_{11}(1-p_{11}) - \ln p_{21}(1-p_{21})); \quad (3-24)$$

$$u + u_2 = \frac{1}{2} (\ln p_{11} + \ln p_{21}) = \frac{1}{2} \ln p_{11} p_{21};$$

so that,

$$\begin{aligned} u_2 &= \frac{1}{4} (\ln p_{11} p_{21} - \ln(1-p_{11})(1-p_{21})) \\ &= \frac{1}{4} \ln \left[\frac{p_{11} p_{21}}{(1-p_{11})(1-p_{21})} \right]; \end{aligned} \quad (3-25)$$

and

$$\begin{aligned} u_{12} &= \frac{1}{4} (\ln p_{11} - \ln(1-p_{11}) - \ln p_{21} + \ln(1-p_{21})) \\ &= \frac{1}{4} \ln \left[\frac{p_{11}}{1-p_{11}} \bigg/ \frac{p_{21}}{1-p_{21}} \right] \\ &= \frac{1}{4} \ln \left[\frac{p_{11}(1-p_{21})}{p_{21}(1-p_{11})} \right]. \end{aligned} \quad (3-26)$$

The log-linear model is not the most convenient model for describing this product-binomial situation. The model usually used is the logit model, defining the relative proportions in the rows,

$$L_i = \ln \left(\frac{p_{i1}}{1-p_{i1}} \right); \quad i=1,2.$$

Solving these logits in terms of the "u" parameters,

$$\begin{aligned}
 L_1 &= \ln p_{11} - \ln(1-p_{11}) \\
 &= u + u_1 + u_2 + u_{12} - (u+u_1-u_2-u_{12}) \\
 &= 2u_2 - 2u_{12}.
 \end{aligned}$$

Letting $w = 2u_2$, $w_1 = 2u_{12}$, and $w_2 = -2u_{12}$, the logit model simplifies to

$$L_i = w + w_i; i=1,2 \quad (3-27)$$

with

$$w_1 + w_2 = 0.$$

Under the homogeneity hypothesis, (3-22), the 2×2 table is given in Figure 5.

p	1-p	1
p	1-p	1

Figure 5. 2×2 Table: Homogeneity Hypothesis

For the log-linear model from (3-23) through (3-26),

$$u = \frac{1}{4} (\ln p(1-p) + \ln p(1-p)) = \frac{1}{2} \ln p(1-p)$$

$$u_1 = \frac{1}{4} (\ln p(1-p) - \ln p(1-p)) = 0$$

$$u_2 = \frac{1}{4} \ln \left[\frac{p^2}{(1-p)^2} \right] = \frac{1}{2} \ln(p/(1-p))$$

$$u_{12} = \frac{1}{4} \ln \left[\frac{p(1-p)}{p(1-p)} \right] = \frac{1}{4} \ln 1 = 0.$$

For the logit model from (3-27),

$$w = 2u_2 = \ln \frac{p}{1-p}$$

$$w_1 = 2u_{12} = 0$$

$$w_2 = -2u_{12} = 0$$

$$L_1 = L_2 = w.$$

Therefore, the homogeneity hypothesis corresponds to $H_0: u_1 = u_{12} = 0$ in the log-linear model and $H_0: w_1 = w_2 = 0$ in the logit model. The convenience of the logit model is obvious. In general, the logit model is preferred in most "stratified" sampling situations where all but one set of marginals are fixed.

These hypotheses can be extended to higher-way tables, although it wasn't until Bartlett (1935) that a precise definition of "no higher-way" interaction in a three-way table was given. In the three-way table the hypothesis of "no second-order" interaction as defined by Bartlett (1935) for a $2 \times 2 \times 2$ table is

$$H_0: p_{111}p_{122}p_{212}p_{221} = p_{112}p_{121}p_{211}p_{222}, \quad (3-28)$$

and expanded to a general $r \times s \times t$ table by Roy and Kastenbaum (1956) is

$$H_0: p_{rst}p_{ijt}p_{rsk}p_{rjk} = p_{ist}p_{rjt}p_{rsk}p_{ijk}; \quad (3-29)$$

$$i=1,2,\dots,(r-1); j=1,2,\dots,(s-1); k=1,2,\dots,(t-1).$$

Considering the $2 \times 2 \times 2$ case, the log-linear model can be parameterized as in (3-15) with

$$\begin{aligned} u_1 &= u_{1(1)} \\ u_2 &= u_{2(1)} \\ u_3 &= u_{3(1)} \\ u_{12} &= u_{12(11)} \\ u_{13} &= u_{13(11)} \\ u_{23} &= u_{23(11)} \\ u_{123} &= u_{123(111)}; \end{aligned} \quad (3-30)$$

so that, for example,

$$\ln p_{111} = u + u_1 + u_2 + u_3 + u_{12} + u_{13} + u_{23} + u_{123}.$$

Using matrix notation,

$$\ln p = X \beta, \quad (3-31)$$

where in lexicographic order

$$(\underline{\ell n p})' = (\ell n p_{111}, \ell n p_{112}, \dots, \ell n p_{222}),$$

$$\underline{\beta}' = (u, u_1, u_2, u_3, u_{12}, u_{13}, u_{23}, u_{123}),$$

and

$$\underline{X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \end{bmatrix}.$$

Note that by factoring $\sqrt{8}$ out of \underline{X} the result is an orthogonal matrix \underline{X}_1 ,

$$\underline{X} = \sqrt{8} \underline{X}_1;$$

so that,

$$\underline{\ell n p} = \sqrt{8} \underline{X}_1 \underline{\beta}, \quad (3-32)$$

and solving for $\underline{\beta}$,

$$\underline{\beta} = \frac{1}{\sqrt{8}} \underline{X}_1' \underline{\ell n p} = \frac{1}{8} \underline{X} \underline{\ell n p}. \quad (3-33)$$

Therefore, in particular,

$$\begin{aligned}
 u_{123} &= \frac{1}{8} (\ln p_{111} - \ln p_{112} - \ln p_{121} + \ln p_{122} - \ln p_{211} \\
 &\quad + \ln p_{212} + \ln p_{221} - \ln p_{222}) \\
 &= \frac{1}{8} \ln \left(\frac{p_{111} p_{122} p_{212} p_{221}}{p_{112} p_{121} p_{211} p_{222}} \right). \quad (3-34)
 \end{aligned}$$

From (3-28) and (3-34) the no second-order interaction hypothesis is clearly equivalent to

$$H_0: \mu_{123} = 0.$$

Product-multinomial sampling in three-way and higher tables allows for several homogeneity-type hypotheses depending on whether one-way, two-way, or higher marginals are fixed, and how many of these marginals are fixed. One of the most common situations involves a series of two-way tables with fixed sample size.

Consider a $2 \times 2 \times 2$ contingency table with one-way marginals $x_{1..}$ fixed. One appropriate hypothesis is to test for homogeneity of the two 2×2 tables formed for each i ,

$$H_0: p_{1jk} = p_{2jk}; j=1,2,; k=1,2. \quad (3-35)$$

These 2×2 tables are given in Figure 6.

		k				k	
		1	2			1	2
j	1	p_{111}	p_{112}	j	1	p_{211}	p_{212}
	2	p_{121}	p_{122}		2	p_{221}	p_{222}
		i=1		1		i=2	
						1	

Figure 6. $2 \times 2 \times 2$ Table: Fixed One-Way Marginals ($x_{1..}$)

Under this hypothesis the solutions, (3-32), for the u_{123} , u_{13} , u_{12} , u_1 terms of the log-linear model are

$$u_{123} = \frac{1}{8} \ln \left(\frac{p_{111} p_{122} p_{212} p_{221}}{p_{112} p_{121} p_{211} p_{222}} \right) = 0$$

$$u_{13} = \frac{1}{8} \ln \left(\frac{p_{111} p_{121} p_{212} p_{222}}{p_{112} p_{122} p_{211} p_{221}} \right) = 0$$

$$u_{12} = \frac{1}{8} \ln \left(\frac{p_{111} p_{112} p_{221} p_{222}}{p_{121} p_{122} p_{211} p_{212}} \right) = 0$$

$$u_1 = \frac{1}{8} \ln \left(\frac{p_{111} p_{112} p_{121} p_{122}}{p_{211} p_{212} p_{221} p_{222}} \right) = 0.$$

Conversely, given $u_{123} = u_{13} = u_{12} = u_1 = 0$,

$$p_{111} p_{122} p_{212} p_{221} = p_{112} p_{121} p_{211} p_{222} \quad (3-36)$$

$$p_{111} p_{121} p_{212} p_{222} = p_{112} p_{122} p_{211} p_{221} \quad (3-37)$$

$$p_{111} p_{112} p_{221} p_{222} = p_{121} p_{122} p_{211} p_{212} \quad (3-38)$$

$$p_{111}p_{112}p_{121}p_{122} = p_{211}p_{212}p_{221}p_{222}. \quad (3-39)$$

Substituting for p_{222} in (3-37) from (3-36),

$$p_{111}p_{121}p_{212} \left(\frac{p_{111}p_{122}p_{212}p_{221}}{p_{112}p_{121}p_{211}} \right) = p_{112}p_{122}p_{211}p_{221};$$

so that,

$$p_{111}p_{212} = p_{112}p_{211}. \quad (3-40)$$

Substituting for p_{221} in (3-38) from (3-37),

$$p_{111}p_{112}p_{222} \left(\frac{p_{111}p_{121}p_{212}p_{222}}{p_{112}p_{122}p_{211}} \right) = p_{121}p_{122}p_{211}p_{212};$$

so that

$$p_{111}p_{222} = p_{122}p_{211}. \quad (3-41)$$

Substituting for p_{222} in (3-36) from (3-38),

$$p_{111}p_{122}p_{212}p_{221} = p_{112}p_{121}p_{211} \left(\frac{p_{121}p_{122}p_{211}p_{212}}{p_{111}p_{112}p_{221}} \right);$$

$$p_{111}p_{221} = p_{121}p_{211}. \quad (3-42)$$

Combining (3-40), (3-41), and (3-42),

$$p_{111}(p_{212}+p_{221}+p_{222}) = p_{211}(p_{112}+p_{121}+p_{122}). \quad (3-43)$$

Since the marginals $x_{1..}$ and $x_{2..}$ are fixed, the corresponding marginal probabilities, $p_{1..}$, $p_{2..}$, equal one. Substituting into (3-43),

$$p_{111}(1-p_{211}) = p_{211}(1-p_{111})$$

$$p_{111} - p_{111}p_{211} = p_{211} - p_{211}p_{111};$$

so that,

$$p_{111} = p_{211}.$$

Similar calculations are easily performed to verify that $p_{1jk} = p_{2jk}$ for all j and k . Therefore,

$$H_0: u_{123} = u_{12} = u_{13} = u_1 = 0 \quad (3-44)$$

is equivalent to (3-35).

The two tests described above (second-order interaction and homogeneity) are the highest order tests possible under the given sampling constraints. In general, a hierarchical set of hypothesis tests

similar to procedures for stepwise regression can be performed on any collection of data. Birch (1963) in his presentation of the log-linear model was one of the first to describe this hierarchical structure. Goodman (1971a) presented an excellent discussion of this structure, along with the appropriate hypothesis tests, relating the procedures to several techniques of stepwise regression. Each of these tests corresponds to the absence of certain "u terms" of the log-linear model. With this log-linear model the interpretation of these tests become relatively simple.

Besides "no second-order" interaction under multinomial sampling for three-way and higher tables, other independence-type hypotheses include pairwise, conditional, partial, mutual, multiple and complete independence. Of particular interest in this thesis is complete independence.

Complete independence specifies that all variables are independent at all levels. It should seem obvious that this would require all interaction terms of the log-linear model to be equal to zero. To demonstrate this, consider the classic sense of complete independence of three variables A, B, and C,

$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

For a three-way contingency table this corresponds to

$$H_0: P_{ijk} = P_{i..}P_{.j.}P_{..k}; i=1,2,\dots,r; j=1,2,\dots,s; k=1,2,\dots,t. \quad (3-45)$$

In particular, consider the $2 \times 2 \times 2$ table. Using the matrix solution, (3-33), of the log-linear model and the null complete independence hypothesis, (3-45),

$$\begin{aligned} u_{123} &= \frac{1}{8} \ln \left(\frac{p_{111}p_{122}p_{212}p_{221}}{p_{112}p_{121}p_{211}p_{222}} \right) \\ &= \frac{1}{8} \ln \left(\frac{p_{1..}p_{.1.}p_{..1}p_{1..}p_{.2.}p_{..2}p_{2..}p_{.1.}p_{..2}p_{2..}p_{.2.}p_{..1}}{p_{1..}p_{.1.}p_{..2}p_{1..}p_{.2.}p_{..1}p_{2..}p_{.1.}p_{..1}p_{2..}p_{.2.}p_{..2}} \right) \\ &= 0. \end{aligned}$$

Similarly, it can be shown that $u_{12} = u_{13} = u_{23} = 0$. Conversely, given $u_{123} = u_{12} = u_{13} = u_{23} = 0$, the relationship in (3-45) can be verified for all i, j, k . Thus, $H_0: u_{123} = u_{12} = u_{13} = u_{23} = 0$ is equivalent to the complete independence hypothesis (3-45).

3.4 Maximum Likelihood Estimates

Maximum likelihood estimates (MLE) as discussed in Chapter II are theoretically satisfactory for analyzing contingency tables, although other best asymptotically normal (BAN) estimates, as described by Neyman (1949), have similar properties. For contingency tables MLE are of two types: those that can be computed directly and those that require some iterative scheme. Whether MLE can be computed directly or must be computed iteratively depends upon the null hypothesis to be tested and the size of the contingency table.

Considering the multinomial sampling distribution, the maximum likelihood estimates for the two-way $r \times s$ table under the hypothesis

of independence can easily be derived. Following the usual procedure, the multinomial density for each cell will be used instead of the joint likelihood density. The kernels are equivalent, and both lead to the unique MLE.

The multinomial density for each cell of a two-way $r \times s$ table was given in (3-2),

$$f(x_{ij}) = N! N^{-r} \prod_{i=1}^r \prod_{j=1}^s m_{ij}^{x_{ij}} x_{ij}! ;$$

$$i=1,2,\dots,r; j=1,2,\dots,s;$$

with multinomial sampling constraint,

$$\sum_{i=1}^r \sum_{j=1}^s m_{ij} = \sum_{i=1}^r \sum_{j=1}^s x_{ij} = N.$$

Taking the logarithm and using a Lagrangian multiplier (λ) to account for the constraint, the log-likelihood function is

$$L = \ln(N!) - N \ln N - \ln(x_{ij}!) + \sum_i \sum_j \ln m_{ij} + \lambda (\sum_i \sum_j m_{ij} - N). \quad (3-46)$$

Under the hypothesis of independence $u_{12(ij)} = 0$, and from (3-13) the log-linear model is

$$\ln m_{ij} = u + u_{1(i)} + u_{2(j)}; i=1,2,\dots,r; j=1,2,\dots,s;$$

or equivalently,

$$\begin{aligned} m_{ij} &= \exp(u + u_{1(i)} + u_{2(j)}) \\ &= \exp u \exp u_{1(i)} \exp u_{2(j)}. \end{aligned}$$

Substituting into (3-46),

$$\begin{aligned} L &= \ln N! - N \ln N - \ln(x_{ij}!) + \sum_i \sum_j x_{ij} (u + u_{1(i)} + u_{2(j)}) \\ &\quad + \lambda \sum_i \sum_j (\exp u \exp u_{1(i)} \exp u_{2(j)} - N). \end{aligned} \quad (3-47)$$

The maximum likelihood estimates, $\hat{m}_{ij} = \exp(\hat{u} + \hat{u}_{1(i)} + \hat{u}_{2(j)})$, can now be derived by taking partial derivative of (3-47) with respect to the variables u , $u_{1(i)}$, $u_{2(j)}$, and λ and setting them equal to zero:

$$\frac{\partial L}{\partial u} = N + \lambda \exp \hat{u} \sum_i \exp \hat{u}_{1(i)} \sum_j \exp \hat{u}_{2(j)} = 0 \quad (3-48)$$

$$\begin{aligned} \frac{\partial L}{\partial u_{1(i)}} &= x_{i\cdot} + \lambda \exp \hat{u} \exp \hat{u}_{1(i)} \sum_j \exp \hat{u}_{2(j)} = 0; \\ &\quad i=1, 2, \dots, r \end{aligned} \quad (3-49)$$

$$\begin{aligned} \frac{\partial L}{\partial u_{2(j)}} &= x_{\cdot j} + \lambda \exp \hat{u} \exp \hat{u}_{2(j)} \sum_i \exp \hat{u}_{1(i)} = 0; \\ &\quad j=1, 2, \dots, 2 \end{aligned} \quad (3-50)$$

$$\frac{\partial L}{\partial \lambda} = \exp \hat{u} \sum_i \exp \hat{u}_{1(i)} \sum_j \exp \hat{u}_{2(j)} - N = 0. \quad (3-51)$$

From (3-48) and (3-51)

$$N + \lambda N = 0,$$

which implies that

$$\lambda = -1.$$

Substituting into (3-49) and (3-50),

$$x_{i.} = \exp \hat{u} \exp \hat{u}_{1(i)} \sum_j \exp \hat{u}_{2(j)}; i=1,2,\dots,r$$

$$x_{.j} = \exp \hat{u} \exp \hat{u}_{2(j)} \sum_i \exp \hat{u}_{1(i)}; j=1,2,\dots,s.$$

Then

$$\begin{aligned} \frac{x_{i.} x_{.j}}{N} &= \frac{\exp \hat{u} \exp \hat{u}_{1(i)} \sum_j \exp \hat{u}_{2(j)} \exp \hat{u} \exp \hat{u}_{2(j)} \sum_i \exp \hat{u}_{1(i)}}{\exp \hat{u} \sum_i \exp \hat{u}_{1(i)} \sum_j \exp \hat{u}_{2(j)}} \\ &= \exp \hat{u} \exp \hat{u}_{1(i)} \exp \hat{u}_{2(j)}. \end{aligned}$$

Therefore, the MLE are

$$\hat{m}_{ij} = x_{i.} x_{.j} / N; i=1,2,\dots,r; j=1,2,\dots,s. \quad (3-52)$$

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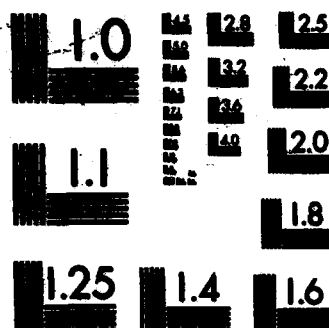
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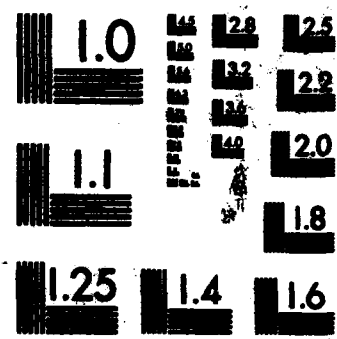
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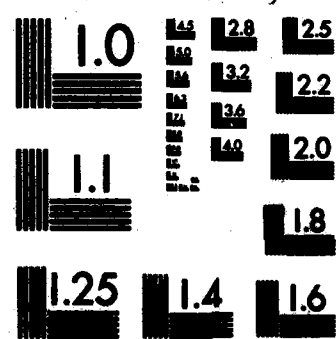
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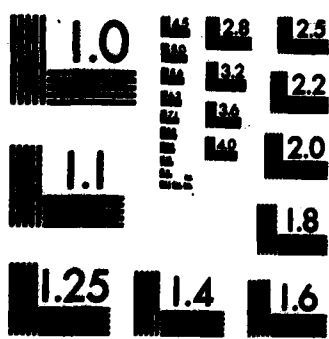
MICROCOPY RESOLUTION TEST CHART
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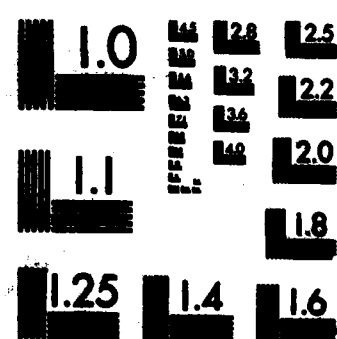
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Under the independence hypothesis these maximum likelihood estimates are unique for the three sampling models, Poisson, multinomial, and product-multinomial [e.g., see Birch (1963), Bishop (1969a), and Haberman (1974b)]. As discussed by Goodman (1971a) and shown by Haberman (1974a), the MLE are unique under any of the Birch hierarchical hypotheses.

As has been shown, the two-way table permits a direct closed-form formula for the maximum likelihood estimates under the independence hypothesis. For higher-way tables and the hypothesis of no "higher-order" interaction this is not the case. Consider the $r \times s \times t$ three-way table and the log-linear model under multinomial sampling,

$$\ln m_{ijk} = u' + u_1(i) + u_2(j) + u_3(k) + u_{12}(ij) + u_{13}(ik) \\ + u_{23}(jk) + u_{123}(ijk);$$

$$i=1,2,\dots,r; j=1,2,\dots,s; k=1,2,\dots,t. \quad (3-53)$$

The hypothesis of no second-order interaction sets $u_{123}(ijk)$ equal to zero for all i, j , and k . Under this hypothesis there is no direct formula to calculate the MLE. There are two iterative procedures currently used, Iterative Proportional Fitting (IPF) and a Newton-Raphson (N-R) procedure. Both procedures converge to the unique MLE for hierarchical hypotheses. Haberman (1974) extensively discusses convergence rates and the other advantages and disadvantages of each procedure. This study will use the IPF procedure.

For the two-way table under the hypothesis of independence, the maximum likelihood estimates were derived above and given in (3-52). Summing (3-52) over i ,

$$\hat{m}_{.j} = x_{..}x_{.j}/N = x_{.j},$$

and summing (3-52) over j ,

$$\hat{m}_{i.} = x_{i.}x_{..}/N = x_{i.}.$$

Note that the maximum likelihood one-way marginals will equal the observed one-way marginals. If this was known in advance, the MLE could be derived from these "sufficient" statistics by fitting the table of MLE to the observed marginals in an iterative fashion. Begin by assuming initial values, $\{\hat{m}_{ij}^{(0)}\}$, for the estimates. Then, use the following formula to fit the table to the observed marginals $x_{i.}$:

$$\hat{m}_{ij}^{(1)} = \hat{m}_{ij}^{(0)} \cdot x_{i.}/\hat{m}_{i.}^{(0)}.$$

Note that

$$\hat{m}_{i.}^{(1)} = \sum_j \hat{m}_{ij}^{(1)} = \sum_j \hat{m}_{ij}^{(0)} x_{i.}/\hat{m}_{i.}^{(0)} = \hat{m}_{i.}^{(0)} x_{i.}/\hat{m}_{i.}^{(0)} = x_{i.}.$$

Next, fit the table to the observed marginals $x_{.j}$ with the formula

$$\hat{m}_{ij}^{(2)} = \hat{m}_{ij}^{(1)} x_{.j} / \hat{m}_{.j}^{(1)}.$$

Note that

$$\hat{m}_{.j}^{(2)} = \sum_i \hat{m}_{ij}^{(2)} = \sum_i \hat{m}_{ij}^{(1)} x_{.j} / \hat{m}_{.j}^{(1)} = \hat{m}_{.j}^{(1)} x_{.j} / \hat{m}_{.j}^{(1)} = x_{.j}.$$

Now calculate $\hat{m}_{ij}^{(2)}$ with initial estimates, $\hat{m}_{ij}^{(0)} = 1$, for all i and j ,

$$\hat{m}_{ij}^{(1)} = 1 \cdot x_{i.} / s = x_{i.} / s$$

$$\hat{m}_{.j}^{(1)} = \sum_i x_{i.} / s = x_{..} / s = N / s$$

$$\hat{m}_{ij}^{(2)} = (x_{i.} / s) (x_{.j} / (N / s)) = x_{i.} x_{.j} / N,$$

which is the exact closed-formed formula, (3-52), previously derived.

For the three-way $r \times s \times t$ contingency table, formulas for the no second-order interaction hypothesis are:

$$\hat{m}_{ijk}^{(1)} = \hat{m}_{ijk}^{(0)} (x_{ij.} / \hat{m}_{ij.}^{(0)})$$

$$\hat{m}_{ijk}^{(2)} = \hat{m}_{ijk}^{(1)} (x_{i.k} / \hat{m}_{i.k}^{(1)}) \quad (3-54)$$

$$\hat{m}_{ijk}^{(3)} = \hat{m}_{ijk}^{(2)} (x_{.jk} / \hat{m}_{.jk}^{(2)}).$$

However, as previously stated, no closed-formed solution exists. The three-step cyclic process above would have to be repeated until some desired accuracy is obtained. As many authors have shown [e.g., Brown

(1959), Fienberg (1972a,b), and Haberman (1974a)] these estimates will converge to the unique maximum likelihood estimates.

For the three-way $r \times s \times t$ table under complete independence, the one-way marginals provide the sufficient statistics to determine uniquely the MLE. The closed-form formula follows the form of the hypothesis (3-45),

$$\hat{m}_{ijk} = x_{i..}x_{.j.}x_{..k}; i=1,2,\dots,r; j=1,2,\dots,s; k=1,2,\dots,t. \quad (3-55)$$

3.5 Pearson Chi-Square (X^2)

Over the years the Pearson chi-square statistic,

$$X^2 = \sum_i (x_i - m_i)^2 / m_i, \quad (3-56)$$

where the m_i are the expected values under the null hypothesis, and the sum is taken over all combinations of categories (cells for a contingency table), has been the most popular statistic for testing a number of different types of hypotheses. As used with contingency tables, the practice has been to replace the m_i with their maximum likelihood estimates (MLE) under the particular hypothesis. Another procedure is to minimize X^2 with respect to the m_i , consistent with the given hypothesis. Neyman (1949) showed that this procedure yielded BAN estimates with properties similar to the MLE. The problem with the procedure is its computational complexity.

For example, consider the two-way 2×2 table under multinomial sampling and the hypothesis of independence. Reparameterizing the

$\{p_{ij}\}$ table to account for the sampling procedure and the independence hypothesis, let $p = p_{1.}$ and $q = p_{.1}$, then $p_{2.} = 1-p$ and $p_{.2} = 1-q$.

Under the independence hypothesis

$$H_0: p_{ij} = p_{i.}p_{.j}; i=1,2; j=1,2;$$

and the multinomial table in Figure 7

p_{11}	p_{12}	$p_{1.}$
p_{21}	p_{22}	$p_{2.}$
$p_{.1}$	$p_{.2}$	1

Figure 7. 2×2 Table: Multinomial Sampling

becomes the reparameterized table in Figure 8.

pq	$p(1-q)$	p
$q(1-p)$	$(1-p)(1-q)$	$1-p$
q	$1-q$	1

Figure 8. 2×2 Table: Independence Hypothesis Reparameterized

Recalling that $m_{ij} = Np_{ij}$ for multinomial sampling, the Pearson statistic, (3-56), becomes

$$X^2 = \sum_i \sum_j (x_{ij} - Np_{ij})^2 / Np_{ij}. \quad (3-57)$$

Substituting the reparameterized p_{ij} into (3-57),

$$X^2 = \frac{(x_{11} - Npq)^2}{Npq} + \frac{(x_{12} - Np(1-q))^2}{Np(1-q)} + \frac{(x_{21} - N(1-p)q)^2}{N(1-p)q} + \frac{(x_{22} - N(1-p)(1-q))^2}{N(1-p)(1-q)}. \quad (3-58)$$

To solve for the \hat{m}_{ij} which minimize X^2 , take partial derivatives of (3-58) with respect to p and q and set equal to zero. Solve these equations for the estimates (\hat{p}, \hat{q}) of (p, q) . These can then be converted with the reparameterized 2×2 table, Figure 8, to estimates \hat{p}_{ij} of p_{ij} and used to find $\hat{m}_{ij} = N\hat{p}_{ij}$. Unfortunately, the solution of these equations requires solving a fifth-order polynomial. Larger tables would require even more difficult computations.

The simplicity of the MLE as well as their good asymptotic properties make them the choice for most situations. In this study the MLE will be used with the Pearson X^2 statistic.

3.6 Minimum Discrimination Information (MDI)

3.6.1 General Derivation

In general, the minimum discrimination information (MDI) approach to the analysis of contingency tables uses a measure of information developed by Kullback (1959), and minimizes this measure with respect to the hypothesis of concern. The measure is then used as a statistic

to test the null hypothesis. Using notation from Gokhale and Kullback (1978), let $p(\omega)$ and $\pi(\omega)$ be the probabilities associated with two contingency tables defined over the set of cells Ω , where $\sum_{\Omega} p(\omega) = \sum_{\Omega} \pi(\omega) = 1$ (corresponding to the multinomial sampling model). The discrimination information between \underline{p} and $\underline{\pi}$ is defined as

$$I(\underline{p}:\underline{\pi}) = \sum_{\Omega} p(\omega) \ln (p(\omega)/\pi(\omega)), \quad (3-59)$$

where $\underline{\pi}$ is fixed and depends on the particular problem, and \underline{p} ranges over a family of distributions, P , which relate to the hypothesis of concern. Kullback (1959) proves several useful properties of this measure. In particular,

$$\text{nonnegativity} - I(\underline{p}:\underline{\pi}) \geq 0 \text{ and } = 0 \text{ if and only if } \underline{p} = \underline{\pi}; \quad (3-60a)$$

$$\text{convexity} - I(\underline{p}:\underline{\pi}) \text{ is a convex function of } \underline{p}; \quad (3-60b)$$

$$\text{nonsymmetry} - I(\underline{p}:\underline{\pi}) \neq I(\underline{\pi}:\underline{p}) \text{ in general}; \quad (3-60c)$$

and that $I(\underline{p}:\underline{\pi})$ is a measure of the deviation between the distributions.

The principle of MDI estimation is to minimize (3-59) over the family of distributions, P , which satisfy certain linearly independent constraints related to the hypothesis of concern. Continuing with the notation of Gokhale and Kullback (1978a), these constraints are expressed in matrix form,

$$\underline{C} \underline{p} = \underline{\theta}, \quad (3-61)$$

where \underline{C} is an $(n+1) \times \Omega$ "design matrix", \underline{p} is the $\Omega \times 1$ probability matrix (cell probabilities), and $\underline{\theta}$ is an $(n+1) \times 1$ matrix of constraining values. The rank of \underline{C} is $n+1$ (i.e., the rows are linearly independent). The solution can be obtained by use of Lagrangian multipliers. The nonlinear optimization problem is

$$\begin{aligned} \text{minimize} \quad & f(\underline{p}) = \sum_{\Omega} p(\omega) \ln (p(\omega)/\pi(\omega)), \\ \text{subject to} \quad & \underline{C} \underline{p} = \underline{\theta} \\ & \underline{p} \geq \underline{0}. \end{aligned} \quad (3-62)$$

The constraints can be written as

$$\sum_{\Omega} C_k(\omega) p(\omega) = \theta_k; \quad k=0,1,\dots,n. \quad (3-63)$$

Using Lagrangian multipliers, λ_k , (3-62) becomes

$$\begin{aligned} \text{minimize} \quad & F(\underline{p}) = \sum_{\Omega} p(\omega) \ln (p(\omega)/\pi(\omega)) \\ & - \sum_{k=0}^n \lambda_k \left(\sum_{\Omega} C_k(\omega) p(\omega) - \theta_k \right). \end{aligned} \quad (3-64)$$

Taking partial derivatives and equating to zero,

$$\begin{aligned} \frac{\partial F(\underline{p})}{\partial p(\omega)} &= \ln(p^*(\omega)/\pi(\omega)) + p^*(\omega) \cdot \frac{\pi(\omega)}{p^*(\omega)} \cdot \frac{1}{\pi(\omega)} \\ &- \sum_{k=0}^n \lambda_k C_k(\omega) = 0, \quad \text{all } \omega \in \Omega \end{aligned} \quad (3-65)$$

and

$$\frac{\partial F(p)}{\partial \lambda_k} = \sum_{\Omega} C_k(\omega) p^*(\omega) - \theta_k = 0; k=0,1,\dots,n. \quad (3-66)$$

Simplifying (3-65) and (3-66),

$$\ln(p^*(\omega)/\pi(\omega)) = \sum_{k=0}^n \lambda_k C_k(\omega) - 1, \text{ all } \omega \in \Omega, \quad (3-67)$$

and

$$\sum_{\Omega} C_k(\omega) p^*(\omega) = \theta_k; k=0,1,\dots,n. \quad (3-68)$$

In multinomial sampling $\sum_{\Omega} p(\omega) = 1$, and letting $C_0(\omega) = \theta_0 = 1$ for all $\omega \in \Omega$ satisfies the constraint. Now solving (3-67) for the $p^*(\omega)$,

$$p^*(\omega) = e^{-1} \pi(\omega) \exp \sum_{k=0}^n (\lambda_k C_k(\omega)), \text{ all } \omega \in \Omega,$$

and letting $\lambda'_0 = \lambda_0 - 1$,

$$p^*(\omega) = \exp(\lambda'_0 + \lambda_1 C_1(\omega) + \lambda_2 C_2(\omega) + \dots + \lambda_n C_n(\omega)) \pi(\omega),$$

all $\omega \in \Omega$ (3-69)

where the λ_k are to be determined to satisfy the constraints, (3-68).

The solution, (3-69), is of exponential form. Rewriting (3-67),

$$\ln(p^*(\omega)/\pi(\omega)) = \lambda_0' + \lambda_1 C_1(\omega) + \lambda_2 C_2(\omega) + \dots + \lambda_n C_n(\omega), \quad (3-70)$$

and the similarity to Birch's log-linear model is apparent [e.g., the $r \times s$ model given in (3-6)]. Birch (1963) presented his model to represent the factors and multiplicative interactions of the contingency table. On the other hand, Kullback's model is naturally derived based on the theory of minimum discrimination information relating two distributions over the same set.

By the convexity property, (3-60b), Equation (3-69) will indeed provide a minimum $I(\underline{p}:\underline{\pi})$; i.e.,

$$I(\underline{p}^*:\underline{\pi}) = \sum_{\Omega} p^*(\omega) \ln(p^*(\omega)/\pi(\omega)) = \min(I(\underline{p}:\underline{\pi}) : \underline{p} \in P). \quad (3-71)$$

This leads to another significant property of the MDI statistic proved by Kullback (1959), the additive property,

$$I(\underline{p}:\underline{\pi}) = I(\underline{p}^*:\underline{\pi}) + I(\underline{p}:\underline{p}^*), \quad \forall \underline{p} \in P. \quad (3-72)$$

3.6.2 The Internal Constraints Problem (ICP)

As explained in Section 3.3, the ultimate goal in contingency table analysis of determining the relationship among the variables can be accomplished by using hypothesis tests on observed tables to fit log-linear models. For the "independence" hypotheses the fitted models depend on some set of observed marginals. Gokhale and Kullback (1978a) define these type of problems as Internal Constraints Problems (ICP). For the "homogeneity" hypotheses no set of marginals

are sufficient to described the model, even though, under product-multinomial sampling, the fitted table may conserve some of the marginals. Gokhale and Kullback (1978) classify these problems into a larger set called External Constraints Problems (ECP). Under the MDI analysis each of these type problems (ICP and ECP) has a different approach. This section will only present the ICP approach.

In considering the observed contingency table and fitting it to a log-linear model, it is more convenient to work in the frequency domain. The MDI estimates (\underline{m}^*) of the expected values (\underline{m}) are $m^*(\omega) = Np^*(\omega)$ for multinomial sampling. For the ICP problems these MDI estimates (\underline{m}^*) are exactly equal to the maximum likelihood estimates (\underline{m}) discussed in Section 3.4. Iterative procedures for obtaining these estimates, similar to Equations (3-54), given for the MLE and the no second-order interaction hypothesis in a three-way table, were developed by Ireland and Kullback (1968a) and Ku and Kullback (1968).

For the ICP problems the MDI statistic (MDIS) for testing hypotheses is

$$2I(\underline{x}:\underline{m}^*) = 2 \sum_{\Omega} x(\omega) \ln(x(\omega)/m^*(\omega)), \quad (3-73)$$

which has an asymptotic chi-square distribution under the null hypothesis. It is interesting to note that this statistic is minus twice the likelihood ratio statistic, whose properties have been empirically compared to the Pearson X^2 and other statistics in some hypothesis testing situations as extensively reviewed in Chapter II.

Working now in the frequency domain for contingency tables, \underline{x} is the $\Omega \times 1$ matrix of observed cells, $x(\omega)$, arranged in lexicographic order. Let \underline{T} be a $\Omega \times (n+1)$ "design matrix" of rank $r+1 \leq \Omega$ ($\underline{T}' = \underline{C}$ in (3-61)). The linearly independent, $n+1$ columns of \underline{T} are $T_i(\omega)$ ($1 \leq \omega \leq \Omega$) for each i , $0 \leq i \leq n$. The "design matrix" corresponds to the fitted observed marginals which, in turn, correspond to the hypothesis of concern. These appropriate observed marginals are the constraints as specified in (3-61). Therefore,

$$\underline{T}'\underline{m}^* = \underline{C}\underline{Np}^* = \underline{N}\underline{\theta} = \underline{C}\underline{x} = \underline{T}'\underline{x}. \quad (3-74)$$

The columns of \underline{T} will be indicator functions for the required fixed marginals. The $T_i(\omega)$ will be one or zero depending on whether or not the cell is included in the corresponding marginal. By convention, for multinomial sampling $T_0(\omega) = 1$ for all ω , and the total sum constraint, $\sum_{\Omega} m^*(\omega) = \sum_{\Omega} x(\omega) = N$, is satisfied.

Changing notation ($\tau_0 = \lambda'_0$, $\tau_i = \lambda'_i$; $1 \leq i \leq n$) and rewriting (3-69) with $m^*(\omega) = Np^*(\omega)$, the MDI estimates have the exponential form,

$$m^*(\omega) = \exp(\tau_0 + \tau_1 T_1(\omega) + \tau_2 T_2(\omega) + \dots + \tau_n T_n(\omega)) N \pi(\omega),$$

$$\text{all } \omega \in \Omega. \quad (3-75)$$

Summing the $m^*(\omega)$ over Ω ,

$$\sum_{\Omega} m^*(\omega) = N \exp(\tau_0) \sum_{\Omega} \exp(\tau_1 T_1(\omega) + \tau_2 T_2(\omega) + \dots + \tau_n T_n(\omega)) \pi(\omega) = N,$$

and solving for τ_0 ,

$$\tau_0 = -\ln \left[\sum_{\Omega} \exp(\tau_1 T_1(\omega) + \tau_2 T_2(\omega) + \dots + \tau_n T_n(\omega)) \pi(\omega) \right]. \quad (3-76)$$

The remaining parameters (τ_i) need to be determined so that the marginal constraints in (3-74) are satisfied for the resulting estimates (\underline{m}) of (3-75).

The iterative proportional fitting procedures, discussed in Section 3-4 for MLE and given in Ireland and Kullback (1968a) and Ku and Kullback (1968) for MDI estimates, cycles through the marginals corresponding to the constraints of (3-74) until the desired level of accuracy is obtained. The distribution $\pi(\omega)$ is used to begin the procedure. The procedure allows $\pi(\omega)$ to be any distribution which satisfies some of the marginal constraints of (3-74) but no others. Any additional marginal constraints satisfied will remain at the end of the cycling process, and the MDI estimates (\underline{m}^*) will contain additional exponential parameters in (3-74) [due to $\pi(\omega)$], which are not related to the Lagrangian parameters corresponding to the hypothesis of concern. It is convenient to let $\pi(\omega)$ be the uniform distribution, $1/\Omega$, for multinomial sampling problems. Then $\pi(\omega)$ satisfies the probability distribution constraint, $\sum_{\Omega} \pi(\omega) = 1$. Correspondingly, the beginning cycle estimates are N/Ω and satisfy the sample size constraint, $\sum_{\Omega} N/\Omega = N$. No other constraint is satisfied, so no extraneous parameters will be introduced into (3-74).

3.6.3 The MDI Log-Linear Model and Hypothesis Tests

The exponential form for $m^*(\omega)$, (3-75), can be changed to a log-linear model by dividing by $N\pi(\omega)$ and taking logarithms,

$$\ln(m^*(\omega)/N\pi(\omega)) = \tau_0 + \tau_1 T_1(\omega) + \tau_2 T_2(\omega) + \dots + \tau_n T_n(\omega),$$

$$\text{all } \omega \in \Omega \quad (3-77)$$

In matrix notation

$$\underline{\ln(m^*/N\pi)} = \underline{T} \underline{\tau}, \quad (3-78)$$

where $\underline{\ln(m^*/N\pi)}$ is the $\Omega \times 1$ matrix with terms $\ln(m^*(\omega)/N\pi(\omega))$ in lexicographic order, and $\underline{\tau}' = (\tau_0, \tau_1, \tau_2, \dots, \tau_n)$. This model can be compared to Birch's log-linear model discussed in Section 3.2.

Consider the 2×2 contingency table and the log-linear model given in (3-15). This "complete" model in matrix notation is

$$\underline{\ln(m)} = \underline{X} \underline{\beta}, \quad (3-79)$$

where

$$(\underline{\ln(m)})' = (\ln m_{11}, \ln m_{12}, \ln m_{21}, \ln m_{22}),$$

$$\underline{\beta}' = (u', u_1, u_2, u_{12}),$$

and

$$\underline{X} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

With the four independent parameters and only four cells the MLE and MDI estimates will exactly equal the observed values; i.e.,

$$\hat{m}_{ij} = m_{ij}^* = x_{ij}; i=1,2; j=1,2.$$

Therefore,

$$\underline{\ln(x)} = \underline{X} \underline{\beta}. \quad (3-80)$$

The corresponding "complete" log-linear model in the MDI parameterization from (3-78) using $\pi_{ij} = 1/4$ is

$$\underline{\ln(x/N/4)} = \underline{\ln(m^*/N/4)} = \underline{T} \underline{\tau}, \quad (3-81)$$

where

$$\underline{\tau}' = (\tau_0, \tau_1, \tau_2, \tau_3)$$

and

$$\underline{T} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the first three columns of \underline{T} specify constraints, $\underline{T}'\underline{m}^* = \underline{T}'\underline{x}$, where the sample size, $\sum_i \sum_j m_{ij}^* = \sum_i \sum_j x_{ij} = N$, and marginals,

$\sum_j m_{1j}^* = \sum_j x_{1j}$ and $\sum_i m_{i1}^* = \sum_i x_{i1}$, are fixed. The fourth column requires that $m_{11}^* = x_{11}$ and, as stated, uniquely determines that

$m_{ij}^* = x_{ij}$, $i=1,2$; $j=1,2$. Solving for the τ parameters,

$$\underline{\tau} = \underline{T}^{-1} \underline{\ln(\underline{x}/N\pi)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \ln(x_{11}/N/4) \\ \ln(x_{12}/N/4) \\ \ln(x_{21}/N/4) \\ \ln(x_{22}/N/4) \end{bmatrix}$$

$$\tau_0 = \ln x_{22} - \ln(N/4)$$

$$\tau_1 = \ln x_{12} - \ln x_{22}$$

$$\tau_2 = \ln x_{21} - \ln x_{22}$$

$$\tau_3 = \ln x_{11} - \ln x_{12} - \ln x_{21} + \ln x_{22}.$$

(3-82)

An alternative way to derive this parameterization would be to use the same starting log-linear model as Birch but to apply different constraints. From (3-6) Birch's model with τ parameters would be

$$\ln(x_{ij}/n\pi_{ij}) = \tau_0 + \tau_{1(i)} + \tau_{2(j)} + \tau_{3(ij)}; i=1,2; j=1,2.$$

Now, instead of applying the summation constraints on the parameters,

let $\tau_{1(2)} = \tau_{2(1)} = \tau_{3(12)} = \tau_{3(21)} = \tau_{3(33)} = 0$ and $\tau_0 = \ln(x_{11}/n\pi)$.

This allows for the following parameterization,

$$\tau_1 = \tau_{1(1)}, \tau_2 = \tau_{2(2)}, \tau_3 = \tau_{3(11)},$$

and the model given in (3-81) with τ parameters of (3-82).

The τ parameters of (3-82) can be compared to the u parameters of (3-80). From (3-80)

$$\underline{\beta} = \underline{X}^{-1} \underline{\ln(X)} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \ln x_{11} \\ \ln x_{12} \\ \ln x_{21} \\ \ln x_{22} \end{bmatrix}$$

and

$$\begin{aligned} u' &= \frac{1}{4} [\ln x_{11} + \ln x_{12} + \ln x_{21} + \ln x_{22}] \\ u_1 &= \frac{1}{4} [\ln x_{11} + \ln x_{12} - \ln x_{21} - \ln x_{22}] \\ u_2 &= \frac{1}{4} [\ln x_{11} - \ln x_{12} + \ln x_{21} - \ln x_{22}] \\ u_{12} &= \frac{1}{4} [\ln x_{11} - \ln x_{12} - \ln x_{21} + \ln x_{22}]. \end{aligned} \tag{3-83}$$

Equating (3-80) and (3-81),

$$\underline{X} \underline{\beta} = \underline{T} \underline{\tau} - \underline{\ln(N/4)}$$

and

$$\underline{\tau} = \underline{T}^{-1} \underline{X} \underline{\beta} + \underline{T}^{-1} \underline{\ln(N/4)}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} u'_0 \\ u_1 \\ u_2 \\ u_{12} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \ln(N/4) \\ \ln(N/4) \\ \ln(N/4) \\ \ln(N/4) \end{bmatrix};$$

so that,

$$\tau_0 = u'_0 - u_1 - u_2 + u_{12} - \ln(N/4)$$

$$\tau_1 = 2(u_1 - u_{12})$$

$$\tau_2 = 2(u_2 - u_{12})$$

$$\tau_3 = 4u_{12}.$$

(3-84)

The interpretations of the τ parameters differ somewhat from those of the u parameters discussed in Section 3.2. The τ parameters establish the $i=2, j=2$ cell as a comparison cell by fixing the $i=1, j=1$ cell. τ_0 is the normalized comparison value. Then τ_1 measures the log-ratio between being in category one of variable one and category two of variable one within category two of variable two. τ_2 measures the log-ratio between being in category one of variable two and category two of variable two within category two of variable one. τ_3 measures a relation similar to u_{12} , an interaction effect between τ_1 and τ_2 ; that is, after correcting for the normalized standard τ_0 , how far τ_1 and τ_2 differ from the normalized fixed cell; i.e.,

$$\begin{aligned}
\tau_3 &= (\ln x_{11} - \ln N/4) - \tau_0 - \tau_1 - \tau_2 \\
&= \ln x_{11} - \ln N/4 - \ln x_{22} + \ln N/4 - \ln x_{12} \\
&\quad + \ln x_{22} - \ln x_{21} + \ln x_{22} \\
&= \ln x_{11} - \ln x_{12} - \ln x_{21} + \ln x_{22}.
\end{aligned}$$

Even though the single effect u and τ parameters differ, the relations in (3-84) indicate that, except for a constant, u_{12} and τ_3 measure the same "interaction" relation among the cell values. Therefore, as might be expected, the independence hypothesis testing procedures for the two models are similar.

Recall that in Birch's log-linear model the classic hypothesis of independence in a 2×2 table with multinomial sampling is equivalent to testing the hypothesis, $H_0: u_{12} = 0$. In Kullback's model the procedure is to form the \underline{T} matrix based on the fixed sample size and marginals corresponding to the null hypothesis. These marginals are the same sufficient statistics required to find the MDI estimates (or equivalent MLE) in the iterative proportional fitting procedure. For the above hypothesis testing situation these are the marginals $m_{1.}$ and $m_{.j}$. Recalling also that the columns must be independent, note that only one of the marginals from each set of two is required; i.e., $m_{1.}$ or $m_{2.}$ and $m_{.1}$ or $m_{.2}$. For a general $r \times s$ table, $r-1$ marginals from $m_{1.}$ and $s-1$ marginals from $m_{.j}$ would be required. Kullback always selects the lowest ordered marginals; so for the 2×2 case

$$\underline{T} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

corresponding to the constraints from (3-74), $\sum_i \sum_j m_{ij}^* = \sum_i \sum_j x_{ij}$,
 $\sum_j m_{1j}^* = \sum_j x_{1j}$, and $\sum_i m_{i1}^* = \sum_i x_{i1}$. The log-linear model from (3-78)
 with $\pi_{ij} = 1/4$ is

$$\begin{aligned} \ln(m_{11}^*/N/4) &= \tau_0 + \tau_1 + \tau_2 \\ \ln(m_{12}^*/N/4) &= \tau_0 + \tau_1 \\ \ln(m_{21}^*/N/4) &= \tau_0 + \tau_2 \\ \ln(m_{22}^*/N/4) &= \tau_0 \end{aligned} \tag{3-85}$$

Solving for the τ parameters,

$$\begin{aligned} \tau_0 &= \ln m_{22}^* - \ln(N/4) \\ \tau_1 &= \ln m_{12}^* - \ln m_{22}^* \\ \tau_2 &= \ln m_{21}^* - \ln m_{22}^* \end{aligned} \tag{3-86}$$

leaving

$$\begin{aligned} \ln(m_{11}^*/N/4) &= \tau_0 + \tau_1 + \tau_2 \\ &= \ln m_{22}^* - \ln(N/4) + \ln m_{12}^* - \ln m_{22}^* \\ &\quad + \ln m_{21}^* - \ln m_{22}^*, \end{aligned}$$

which implies that

$$\ln m_{11}^* - \ln m_{12}^* - \ln m_{21}^* + \ln m_{22}^* = 0, \quad (3-87)$$

or as might have been expected, $\tau_3 = 0$. The complete model, (3-81), allows the τ parameters, (3-82), to be calculated directly based on the observed values. The independence model, (3-85), uses the MDI estimates, which have been calculated under independence to require that τ_3 be equal to zero as determined in (3-87). Thus, the independence hypothesis is equivalent to

$$H_0: \tau_3 = 0 \text{ in the Kullback model,}$$

or

$$H_0: u_{12} = 0 \text{ in the Birch model.}$$

Equation (3-87) can easily be verified by using the MDI estimates (or MLE), $m_{ij}^* = x_{i.}x_{.j}/N$:

$$\begin{aligned} & \ln(x_{1.}x_{.1}/N) - \ln(x_{1.}x_{.2}/N) - \ln(x_{2.}x_{.1}/N) \\ & \quad + \ln(x_{2.}x_{.2}/N) \\ &= \ln x_{1.} + \ln x_{.1} - \ln N - \ln x_{1.} - \ln x_{.2} \\ & \quad + \ln N - \ln x_{2.} - \ln x_{.1} + \ln N + \ln x_{2.} \\ & \quad + \ln x_{.2} - \ln N \\ &= 0. \end{aligned}$$

Consider now the three-way $2 \times 2 \times 2$ table under multinomial sampling. The complete Kullback log-linear model is

$$\underline{\ln(X/N\pi)} = \underline{T} \underline{\tau}; \quad (3-88)$$

where $\underline{\ln(X/N\pi)}$ is an 8×1 matrix whose values are $\ln(x_{ijk}/N\pi_{ijk})$; $i=1,2; j=1,2; k=1,2$;

$$\underline{\tau}' = (L, \tau_1^i, \tau_1^j, \tau_1^k, \tau_{11}^{ij}, \tau_{11}^{ik}, \tau_{11}^{jk}, \tau_{111}^{ijk});$$

and

$$\underline{T} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note the change in the τ notation. This new notation from Gokhale and Kullback (1978) allows extension to any size table. L is the normalizing term (τ_0). The number of subscripts corresponds to the type effect (one for main effect, two for first-order interaction, and three for second-order interaction). The superscripts designate the variables in the effect. The subscripts designate the category of the corresponding superscript variables. Solving the model for the

parameters (letting $\pi_{ijk} = 1/8$),

$$\begin{aligned}
 L &= \ln(x_{222}/N/8) \\
 \tau_1^i &= \ln(x_{122}/x_{222}) \\
 \tau_1^j &= \ln(x_{212}/x_{222}) \\
 \tau_1^k &= \ln(x_{221}/x_{222}) \\
 \tau_{11}^{ij} &= \ln(x_{112}/x_{222} \cdot x_{222}/x_{122} \cdot x_{222}/x_{212}) = \ln(x_{112}x_{222}/x_{122}x_{212}) \\
 \tau_{11}^{ik} &= \ln(x_{121}/x_{222} \cdot x_{222}/x_{122} \cdot x_{222}/x_{221}) = \ln(x_{121}x_{222}/x_{122}x_{221}) \\
 \tau_{11}^{jk} &= \ln(x_{211}/x_{222} \cdot x_{222}/x_{212} \cdot x_{222}/x_{221}) = \ln(x_{211}x_{222}/x_{212}x_{221}) \\
 \tau_{111}^{ijk} &= \ln(x_{111}/x_{222} \cdot x_{222}/x_{221} \cdot x_{222}/x_{212} \cdot x_{222}/x_{122} \cdot x_{122}x_{212}/ \\
 &\quad x_{112}x_{222} \cdot x_{122}x_{221}/x_{121}x_{222} \cdot x_{212}x_{221}/x_{211}x_{222}) \\
 &= \ln(x_{111}x_{122}x_{212}x_{221}/x_{112}x_{121}x_{211}x_{222}). \quad (3-89)
 \end{aligned}$$

The interpretation of the τ parameters follows from the 2×2 model. Note that the x_{222} term has been established as the comparison or normalization term. The main effects are calculated as log-ratios within the second levels of the other variables. The first-order interaction terms are log-ratios within the second level of the other variable. The remaining term, τ_{111}^{ijk} , is then a measure of the difference in the value of the $i=1, j=1, k=1$ cell and what is predicted by the terms which have been calculated within the $i=2, j=2, k=2$ levels. Note the relation to the u_{123} term of Birch's log linear model given in (3-34),

$$\tau_{111}^{ijk} = 8u_{123}.$$

τ_{111}^{ijk} is indeed a measure of second-order interaction under the classic Bartlett definition.

The no second-order interaction model is

$$\ln(\underline{m^*}/N\underline{\pi}) = \underline{T} \underline{\tau}, \quad (3-90)$$

where $\ln(\underline{m^*}/N\underline{\pi})$ is an 8×1 matrix whose values are $\ln(m_{ijk}^*/N\pi_{ijk})$, $i=1,2$; $j=1,2$; $k=1,2$; and m_{ijk}^* are the MDI estimates calculated by iterative proportional fitting;

$$\underline{\tau}' = (L, \tau_1^i, \tau_1^j, \tau_1^k, \tau_{11}^{ij}, \tau_{11}^{ik}, \tau_{11}^{jk});$$

and

$$\underline{T} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The τ terms are the same as in (3-89) with x_{ijk} replacing m_{ijk}^* for all i, j , and k . τ_{111}^{ijk} has then been forced to zero. Therefore, as in

Birch's model ($H_0: u_{123} = 0$), the null hypothesis of no second-order interaction is equivalent to $H_0: \tau_{111}^{ijk} = 0$ in Kullback's log-linear model, (3-88).

3.6.4 Asymptotic Covariances

With respect to Kullback's MDI, some discussion of the asymptotic covariance structure will help to relate the other contingency table methodologies. Kullback (1970) derives a general formula for finding the asymptotic covariance structure of any ICP based on the relationship of the "moment" parameters (observed marginals and observed cells) and the "natural" parameters (τ parameters in the MDI log-linear model). Given the MDI model [(3-78), where $\underline{m}^* = \underline{x}$ for complete models] in matrix notation with $\pi(\omega) = 1/\Omega$ (for all $\omega \in \Omega$), let $\underline{D}_{\underline{x}}$ be a diagonal matrix with diagonal entries, $x(\omega)$, in lexicographic order. Find the matrix

$$\underline{S} = \begin{bmatrix} \underline{S}_{11} & \underline{S}_{12} \\ \underline{S}_{21} & \underline{S}_{22} \end{bmatrix} = \underline{T}' \underline{D}_{\underline{x}} \underline{T}, \quad (3-91)$$

where \underline{S}_{11} is a 1×1 matrix corresponding to the normalizing constraint, $\sum_{\Omega} m^*(\omega) = \sum_{\Omega} x(\omega) = N$, \underline{S}_{22} is a $n \times n$ matrix, and $\underline{S}_{12} = \underline{S}_{21}'$ are $1 \times n$ matrices. Then the estimate of the asymptotic covariance matrix of the moment parameters [those specified by \underline{T} in the constraints of (3-74), $\underline{T}' \underline{m}^* = \underline{T}' \underline{x}$] is

$$\underline{S}_{22.1} = \underline{S}_{22} - \underline{S}_{21} \underline{S}_{11}^{-1} \underline{S}_{12}. \quad (3-92)$$

The corresponding estimate of the covariance matrix of the τ parameters of (3-78) is the inverse matrix $\underline{S}_{22.1}^{-1}$.

To demonstrate the procedure, consider the 2×2 table and complete model, (3-81). Following the above procedure,

$$\underline{S} = \underline{T}' \underline{D} \underline{x} \underline{T} = \begin{bmatrix} N & x_{1.} & x_{.1} & x_{11} \\ x_{1.} & x_{1.} & x_{11} & x_{11} \\ x_{.1} & x_{11} & x_{.1} & x_{11} \\ x_{11} & x_{11} & x_{11} & x_{11} \end{bmatrix} = \begin{bmatrix} \underline{S}_{11} & \underline{S}_{12} \\ \underline{S}_{21} & \underline{S}_{22} \end{bmatrix} \quad (3-93)$$

$$\underline{S}_{22.1} = \underline{S}_{22} - \underline{S}_{21} \underline{S}_{11}^{-1} \underline{S}_{12} = \begin{bmatrix} \frac{x_{1.}x_{2.}}{N} & x_{11} - \frac{x_{1.}x_{.1}}{N} & \frac{x_{11}x_{2.}}{N} \\ x_{11} - \frac{x_{1.}x_{.1}}{N} & \frac{x_{.1}x_{.2}}{N} & \frac{x_{11}x_{.2}}{N} \\ \frac{x_{11}x_{2.}}{N} & \frac{x_{11}x_{.2}}{N} & x_{11} - \frac{x_{11}^2}{N} \end{bmatrix} \quad (3-94)$$

$$\underline{S}_{22.1}^{-1} = \begin{bmatrix} \frac{1}{x_{12}} + \frac{1}{x_{22}} & \frac{1}{x_{22}} & -\frac{1}{x_{12}} - \frac{1}{x_{22}} \\ \frac{1}{x_{22}} & \frac{1}{x_{21}} + \frac{1}{x_{22}} & -\frac{1}{x_{21}} - \frac{1}{x_{22}} \\ -\frac{1}{x_{12}} - \frac{1}{x_{22}} & -\frac{1}{x_{21}} - \frac{1}{x_{22}} & \frac{1}{x_{11}} + \frac{1}{x_{12}} + \frac{1}{x_{21}} + \frac{1}{x_{22}} \end{bmatrix} \quad (3-95)$$

For the complete model $\underline{S}_{22.1}$ gives the estimated covariance matrix for the moment parameters $x_{1.}$, $x_{.1}$, and x_{11} . This can easily

be verified since the x_{ij} are observations from a multinomial distribution. These observations can be used to estimate the p_{ij} , $\hat{p}_{ij} = x_{ij}/N$, and to calculate each term of $S_{22.1}$ from variance/covariance formulas for the multinomial. For example,

$$\begin{aligned}
 V(x_{1.}) &= V(x_{11} + x_{12}) = V(x_{11}) + V(x_{12}) + 2\text{COV}(x_{11}, x_{12}) \\
 &\approx N \hat{p}_{11}(1-\hat{p}_{11}) + N \hat{p}_{12}(1-\hat{p}_{12}) + 2(-N \hat{p}_{11}\hat{p}_{12}) \\
 &= \frac{1}{N} x_{11}(x_{12} + x_{21} + x_{22}) + \frac{1}{N} x_{12}(x_{11} + x_{21} + x_{22}) - \frac{1}{N} 2x_{11}x_{12} \\
 &= \frac{1}{N} x_{11}(x_{21} + x_{22}) + \frac{1}{N} x_{12}(x_{21} + x_{22}) \\
 &= (x_{1.}x_{2.})/N,
 \end{aligned}$$

which is the (1,1) term in (3-94). Other terms are similarly verified.

A more general approach for calculating asymptotic covariance matrices for asymptotic normal parameters is the well-known "Delta Method". Bishop, Fienberg, and Holland (1975) provide theorems (Theorems 14.6-1, 2, 3, 4) for its use in both the single variate and multivariate cases. As applied to contingency tables, to find the estimated asymptotic covariance of a matrix function ($f(\underline{x})$ of size $n \times 1$) of the observations, the function must be differentiable with respect to \underline{x} . This derivative is

$$\left(\frac{\partial f}{\partial \underline{x}} \right)_{1\omega} = \left(\frac{\partial f_1}{\partial x(\omega)} \right) .$$

Next, the estimated covariance matrix, $\underline{V}(\underline{x})$, of the observations from the appropriate sampling distribution must be found. The estimated asymptotic covariance matrix for $\underline{f}(\underline{x})$ is then

$$\underline{\tilde{V}}(\underline{f}(\underline{x})) = (\partial \underline{f} / \partial \underline{x}) \underline{\tilde{V}}(\underline{x}) (\partial \underline{f} / \partial \underline{x})'. \quad (3-96)$$

For the 2×2 table and complete model, (3-81),

$$(\underline{f}(\underline{x}))' = (x_{1.}, x_{.1}, x_{11});$$

so that,

$$\frac{\partial \underline{f}}{\partial \underline{x}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

which is the \underline{I} matrix minus the top row corresponding to the sample size constraint. From the multinomial distribution, the estimated covariance matrix for \underline{x} is

$$\underline{\tilde{V}}(\underline{x}) = \frac{1}{N} \begin{bmatrix} x_{11}(N-x_{11}) & -x_{12}x_{11} & -x_{21}x_{11} & -x_{22}x_{11} \\ -x_{11}x_{12} & x_{12}(N-x_{12}) & -x_{21}x_{12} & -x_{22}x_{12} \\ -x_{11}x_{21} & -x_{12}x_{21} & x_{21}(N-x_{21}) & -x_{22}x_{21} \\ -x_{11}x_{22} & -x_{12}x_{22} & -x_{21}x_{22} & x_{22}(N-x_{22}) \end{bmatrix} \quad (3-97)$$

Then from (3-96)

$$\tilde{V}(\underline{f}(\underline{x})) = \frac{1}{N} \begin{bmatrix} x_{1.}x_{2.} & x_{11}x_{22} - x_{12}x_{21} & x_{11}x_{2.} \\ x_{11}x_{22} - x_{12}x_{21} & x_{.1}x_{.2} & x_{11}x_{.2} \\ x_{11}x_{2.} & x_{11}x_{.2} & x_{11}(N - x_{11}) \end{bmatrix}. \quad (3-98)$$

Expanding terms (1,2) and (2,1) of $\underline{S}_{22.1}$ given in (3-94),

$$\begin{aligned} x_{11} - \frac{x_{1.}x_{.1}}{N} &= [x_{11}(x_{11} + x_{12} + x_{21} + x_{22}) - (x_{11} + x_{12})(x_{11} + x_{22})]/N \\ &= [x_{11}x_{22} - x_{12}x_{21}]/N. \end{aligned}$$

Hence $\underline{S}_{22.1}$ and $\tilde{V}(\underline{f}(\underline{x}))$ are equivalent.

Bishop, Fienberg, and Holland (1975) provide a theorem (14.6-4) for the "Delta Method" directly applicable to log-linear functions, $\underline{f}(\underline{x})$, under multinomial sampling. If $f_i(\underline{x}) = \sum_{\omega} C_i(\omega) \ln x(\omega)$; $i=1,2,\dots,n$; then the terms of the $\partial \underline{f} / \partial \underline{x}$ matrix are

$$\frac{\partial f_i}{\partial x(\omega)} = C_i(\omega) / x(\omega).$$

In matrix notation

$$\partial \underline{f} / \partial \underline{x} = \underline{C} \underline{D}_{\underline{x}}^{-1}, \quad (3-99)$$

where $(\underline{C})_{i\omega} = (C_i(\omega))$. Then the theorem supplies the estimated asymptotic covariance matrix of $\underline{f}(\underline{x})$,

$$\tilde{V}(\underline{f}(\underline{x})) = \underline{C} \underline{D}^{-1}_{\underline{x}} \underline{C}' - (\underline{e} \underline{C}')' \underline{e} \underline{C}', \quad (3-100)$$

where $\underline{e} = (1, 1, \dots, 1)$ is a $1 \times \Omega$ vector.

This procedure can be applied to find the estimated asymptotic covariance of the taus in the 2×2 complete model under multinomial sampling. These taus, (τ_1, τ_2, τ_3) , as given in (3-82) are linear combinations of the logarithms of \underline{x} . Forming the appropriate matrices,

$$\underline{C} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\underline{e} = (1, 1, 1, 1)$$

then

$$\underline{f}(\underline{x}) = \underline{C} \ln(\underline{x}).$$

From (3-100)

$$\tilde{V}(\tau_1, \tau_2, \tau_3) = \tilde{V}(\underline{f}(\underline{x})) = \begin{bmatrix} \frac{1}{x_{12}} + \frac{1}{x_{22}} & \frac{1}{x_{22}} & -\frac{1}{x_{12}} - \frac{1}{x_{22}} \\ \frac{1}{x_{22}} & \frac{1}{x_{21}} + \frac{1}{x_{22}} & -\frac{1}{x_{21}} - \frac{1}{x_{22}} \\ -\frac{1}{x_{12}} - \frac{1}{x_{22}} & -\frac{1}{x_{21}} - \frac{1}{x_{22}} & \frac{1}{x_{11}} + \frac{1}{x_{12}} + \frac{1}{x_{21}} + \frac{1}{x_{22}} \end{bmatrix}, \quad (3-101)$$

and comparing to (3-95), $\underline{V}(\tau_1, \tau_2, \tau_3)$ and $\underline{S}_{22.1}^{-1}$ are identical.

With the above procedure the estimate of the asymptotic covariance matrix for the u terms of Birch's log-linear model can also be calculated. For the 2×2 table, from (3-83) for (u_1, u_2, u_{12})

$$\underline{C} = \frac{1}{4} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

and using (3-100),

$$\begin{aligned} & \underline{\tilde{V}}(u_1, u_2, u_{12}) \\ &= \frac{1}{16} \begin{bmatrix} \frac{1}{x_{11}} + \frac{1}{x_{12}} + \frac{1}{x_{21}} + \frac{1}{x_{22}} & \frac{1}{x_{11}} - \frac{1}{x_{12}} - \frac{1}{x_{21}} + \frac{1}{x_{22}} & \frac{1}{x_{11}} - \frac{1}{x_{12}} + \frac{1}{x_{21}} - \frac{1}{x_{22}} \\ \frac{1}{x_{11}} - \frac{1}{x_{12}} - \frac{1}{x_{21}} + \frac{1}{x_{22}} & \frac{1}{x_{11}} + \frac{1}{x_{12}} + \frac{1}{x_{21}} + \frac{1}{x_{22}} & \frac{1}{x_{11}} + \frac{1}{x_{12}} - \frac{1}{x_{21}} - \frac{1}{x_{22}} \\ \frac{1}{x_{11}} - \frac{1}{x_{12}} + \frac{1}{x_{21}} - \frac{1}{x_{22}} & \frac{1}{x_{11}} + \frac{1}{x_{12}} - \frac{1}{x_{21}} - \frac{1}{x_{22}} & \frac{1}{x_{11}} + \frac{1}{x_{12}} + \frac{1}{x_{21}} + \frac{1}{x_{22}} \end{bmatrix}. \end{aligned}$$

(3-102)

Note that $\underline{\tilde{V}}(\tau_3) = 16V(u_{12})$, which is consistent with the relation in (3-84). The other relations for τ_1 and τ_2 in (3-84) could also easily be verified.

3.6.5 Asymptotic Comparisons

Using these covariances, Kullback has derived various estimators for the MDI statistics. These are derived in general terms in Appendix A and given as Equations (A-21) and (A-22). As an example, consider again the 2×2 table and the ICP problem of the independence hypothesis under multinomial sampling. The appropriate hypothesis testing statistic is given in Equation (3-73), $2I(\underline{x}, \underline{m}^*)$. Here the observed and estimated tables have reversed roles. The partitioning described in Appendix A, (A-18), is

$$\begin{aligned}\underline{\theta}^* &= (\underline{\theta}_A^*, \underline{\theta}_B^*) = ((x_{1.}, x_{.1}), x_{11}) \\ &= (\underline{\theta}_A, \underline{\theta}_B) = ((m_{1.}^*, m_{.1}^*), m_{11}^*),\end{aligned}\quad (3-103)$$

where $\underline{\theta}_A^* = \underline{\theta}_A$. The $\underline{\Sigma}$ matrix is the estimated covariance matrix of the moment parameters $(m_{1.}^*, m_{.1}^*, m_{11}^*)$, which is $\underline{S}_{22.1}$ of (3-94) with the MDI estimates, m_{ij}^* , in place of the x_{ij} , and $\underline{\Sigma}^{-1}$ is $\underline{S}_{22.1}^{-1}$ of (3-95) with m_{ij}^* replacing the x_{ij} . Noting that $\underline{\theta}_B^*$ and $\underline{\theta}_B$ are 1×1 matrices and $\underline{\theta}_A^* = \underline{\theta}_A$, from Equation (A-16) and (A-22),

$$\begin{aligned}2I(\underline{p}^*; \underline{\pi}) &\approx (\underline{\theta}^* - \underline{\theta})' \underline{\Sigma}^{-1} (\underline{\theta}^* - \underline{\theta}) \\ &= (\underline{\theta}_B^* - \underline{\theta}_B)' \underline{\Sigma}_{BB.A}^{-1} (\underline{\theta}_B^* - \underline{\theta}_B).\end{aligned}$$

Then, $\underline{\Sigma}_{BB.A}^{-1}$ must be the (3,3) term of $\underline{S}_{22.1}^{-1}$, $\frac{1}{m_{11}^*} + \frac{1}{m_{12}^*} + \frac{1}{m_{21}^*} + \frac{1}{m_{22}^*}$, and the approximation is

$$2I(\underline{x}:\underline{m}^*) \approx (x_{11} - m_{11}^*)^2 \left(\frac{1}{m_{11}^*} + \frac{1}{m_{12}^*} + \frac{1}{m_{21}^*} + \frac{1}{m_{22}^*} \right). \quad (3-104)$$

Recalling that the marginals of the observed and MDI tables are equal,

$$x_{11} - m_{11}^* = x_{1.} - m_{1.}^* = x_{1.} - x_{12} - (m_{1.}^* - m_{12}^*) = m_{12}^* - x_{12}$$

or

$$(x_{11} - m_{11}^*)^2 = (x_{12} - m_{12}^*)^2;$$

$$x_{11} - m_{11}^* = x_{.1} - x_{21} - (m_{.1}^* - m_{21}^*) = m_{21}^* - x_{21}$$

or

$$(x_{11} - m_{11}^*)^2 = (x_{21} - m_{21}^*)^2;$$

$$x_{11} - m_{11}^* = m_{12}^* - x_{12} = m_{.2}^* - m_{22}^* - (x_{.2} - x_{22}) = x_{22} - m_{22}^*$$

or

$$(x_{11} - m_{11}^*)^2 = (x_{22} - m_{22}^*)^2.$$

Substituting into (3-104),

$$\begin{aligned}
2I(\underline{x}:\underline{m}^*) &\approx (x_{11} - m_{11}^*)^2/m_{11}^* + (x_{12} - m_{12}^*)^2/m_{12}^* \\
&\quad + (x_{2.} - m_{2.}^*)^2/m_{2.}^* + (x_{22} - m_{22}^*)^2/m_{22}^* \\
&= \sum_i \sum_j (x_{ij} - m_{ij}^*)^2/m_{ij}^*. \tag{3-105}
\end{aligned}$$

For the ICP problems the MDI estimates, m_{ij}^* , are equal to the MLE; therefore, the approximation is the Pearson X^2 with MLE for the expected values.

If now $2I(\underline{m}^*:\underline{x})$ is estimated using the same procedure, the covariance matrices, $\underline{\Sigma}$ and $\underline{\Sigma}^{-1}$, are in terms of the observed table, and Equation (A-22) becomes

$$2I(\underline{m}^*:\underline{x}) \approx (x_{11} - m_{11}^*)^2 \left(\frac{1}{x_{11}} + \frac{1}{x_{12}} + \frac{1}{x_{2.}} + \frac{1}{x_{22}} \right). \tag{3-106}$$

Using the same substitutions as above, the right-hand side of (3-106) is equal to

$$\sum_i \sum_j (x_{ij} - m_{ij}^*)^2/x_{ij}.$$

This is Neyman's modified chi-square statistic, Y^2 , with the MLE for estimates of the expected values. This statistic is useful in ECP type problems.

If (A-21) is now used,

$$2I(\underline{x}:\underline{m}^*) \approx \underline{\tau}_B' \underline{\Sigma}_{BB} \underline{A} \underline{\tau}_B,$$

where $\underline{\tau}_B$ is τ_3 . From (3-82), τ_3 is equal to $(\ln x_{11} - \ln x_{12} - \ln x_{21} + \ln x_{22})$, so (A-21) is

$$2I(\underline{x}:\underline{m}^*) \approx (\ln x_{11} - \ln x_{12} - \ln x_{21} + \ln x_{22})^2 \left(\frac{1}{x_{11}} + \frac{1}{x_{12}} + \frac{1}{x_{21}} + \frac{1}{x_{22}} \right)^{-1}. \quad (3-107)$$

In the next section, the right-hand side of (3-107) will be shown to be equal to the GSK statistic for the 2×2 independence hypothesis test.

All four statistics (Pearson, Neyman, Kullback, and GSK) are asymptotically related through their asymptotic covariance structure. The Pearson and GSK statistics are asymptotically equivalent to the Kullback ICP statistic (log-likelihood ratio statistic), and the Neyman statistic is asymptotically equivalent to the Kullback ECP statistic.

3.7 Grizzle, Starmer, and Koch (GSK)

The Grizzle, Starmer, and Koch (GSK) approach to the analysis of contingency tables is based on the methodology of weighted least squares. The approach has gained wide-acceptance and has been applied to many varied types of problems. Probably its most attractive feature is that it can be used with a number of different assumed models to include both the linear and log-linear models. First, the general procedure will be discussed, then the procedure with an assumed log-

linear model will be described.

3.7.1 General Derivation

In order to better compare the contingency table methodologies, Kullback's notation will be used in place of the standard GSK notation. Let

$$\underline{p}' = (p(1), p(2), \dots, p(\Omega)) \quad (3-108)$$

be the $1 \times \Omega$ vector of table probabilities in lexicographic order. Let $\hat{p}(\omega)$ be sample estimates of these probabilities. For example, for multinomial sampling

$$\hat{p}(\omega) = x(\omega)/N, \quad (3-109)$$

where $x(\omega)$ are the observed values and N is the total sample size. Then,

$$\hat{\underline{p}}' = (\hat{p}(1), \hat{p}(2), \dots, \hat{p}(\Omega)) \quad (3-110)$$

$$\underline{x}' = (x(1), x(2), \dots, x(\Omega)).$$

Knowing that the asymptotic covariance of $\hat{\underline{p}}$ is $\underline{V}(\hat{\underline{p}}) = \underline{V}(\underline{p})$, where the form of $\underline{V}(\underline{p})$ is determined from the sampling model, an estimate of $\underline{V}(\underline{p})$ can be obtained by replacing $p(\omega)$ by $\hat{p}(\omega)$ in $\underline{V}(\underline{p})$. Let this estimate be $\tilde{\underline{V}}(\underline{p})$. Let $f_m(\underline{p})$ ($m=1, 2, \dots, n$) be a sequence of functions of the elements of \underline{p} that have partial derivatives with respect to the $p(\omega)$ up to second order. Let

$$f_m(\hat{p}) = f_m(p) \text{ evaluated at } p = \hat{p}$$

$$[F(p)]' = [f_1(p), f_2(p), \dots, f_n(p)]$$

$$F' = [F(\hat{p})]' = [f_1(\hat{p}), f_2(\hat{p}), \dots, f_n(\hat{p})] \quad (3-111)$$

$$H = \left[\frac{\partial f_m(p)}{\partial p(\omega)} \right]_{p(\omega) = \hat{p}(\omega)} \quad (n \times \Omega)$$

$$S = H \tilde{V}(p) H' \quad (n \times n)$$

From the "Delta Method" described in Section 3.6 and Equation (3-96), S is the estimate of the covariance matrix of $F(p)$. If $F(p)$ is a linear function of p , then S is exact in the sense that it is independent of sample size. If $F(p)$ is a non-linear function of p then S is the estimate of the asymptotic covariance matrix.

Next, assuming

$$F(p) = X \beta, \quad (3-112)$$

where X is a known $(n \times d)$ design matrix [analogous to Kullback's C , Equation (3-61) and (3-69), or T , Equation (3-78)] and β is a $(d \times 1)$ vector of unknown parameters [analogous to Kullback's τ , Equation (3-78)], the weighted least squares method determines that a best asymptotic normal [BAN, Neyman (1949)] estimate of β is b , where b is the vector which minimizes

$$x_F^2 = (F - X b)' S^{-1} (F - X b). \quad (3-113)$$

This weighted least squares estimator is

$$\underline{b} = (\underline{X}'\underline{S}^{-1}\underline{X})^{-1}\underline{X}'\underline{S}^{-1}\underline{F}. \quad (3-114)$$

The statistic, χ^2_F , in (3-113) is then a test on the fit of the model, (3-112), and has an asymptotic chi-square distribution with $n-d$ degrees of freedom under the null hypothesis.

If the model does fit, a test can be performed on a linear hypothesis of $\underline{\beta}$,

$$H_0: \underline{K} \underline{\beta} = \underline{0}, \quad \underline{K} \text{ is } (k \times d), \quad (3-115)$$

with the statistic

$$G^2 = (\underline{K} \underline{b})' (\underline{K} (\underline{X}'\underline{S}^{-1}\underline{X})^{-1} \underline{K}')^{-1} \underline{K} \underline{b}, \quad (3-116)$$

which under the null hypothesis has an asymptotic chi-square distribution with k degrees of freedom.

3.7.2 Log-Linear Model and Hypothesis Tests

Now suppose that the structure of $\underline{F}(p)$ is log-linear; i.e.,

$$\underline{F}(p) = \underline{C}(\ln p), \quad (3-117)$$

where \underline{C} is $(n \times \Omega)$. Then from (3-111)

$$\underline{H} = \underline{C} \underline{D}^{-1}_{\hat{p}}, \quad (3-118)$$

where $\underline{D}_{\hat{p}}$ is a diagonal matrix with p along its diagonal. From the "Delta Method" of Section 3.6 and (3-111),

$$\underline{S} = \underline{C} \underline{D}_{\hat{p}}^{-1} \underline{\tilde{V}}(p) \underline{D}_{\hat{p}}^{-1} \underline{C}', \quad (3-119)$$

the estimated asymptotic covariance of $\underline{F}(p)$. Under multinomial sampling, from (3-100) this becomes

$$\underline{S} = N^{-1} [\underline{C} \underline{D}_{\hat{p}}^{-1} \underline{C}' - (\underline{e} \underline{C}')' \underline{e} \underline{C}']. \quad (3-120)$$

With \underline{S} from (3-119) or (3-120), hypothesis tests are performed using (3-112) through (3-116). Covariance formulas for \underline{b} and \underline{F} are

$$\underline{\tilde{V}}(\underline{b}) = (\underline{X}' \underline{S}^{-1} \underline{X})^{-1} \quad (3-121)$$

$$\underline{\tilde{V}}(\underline{F}) = \underline{X} (\underline{X}' \underline{S}^{-1} \underline{X})^{-1} \underline{X}'. \quad (3-122)$$

As an example, consider the 2×2 table under multinomial sampling and the no interaction hypothesis. The first step is to choose an appropriate log-linear model. Consider Birch's complete log-linear model with the parameterization of (3-15) and (u_1, u_2, u_{12}) formulas in (3-83). Forming Equation (3-117),

$$\underline{F}(p) = \underline{C} \underline{\ln} p, \quad (3-123)$$

where

$$\underline{C} = \frac{1}{4} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

and

$$(\underline{\ln p})' = (\ln p_{11}, \ln p_{12}, \ln p_{21}, \ln p_{22}).$$

From (3-112) the model is

$$\underline{F(p)} = \underline{C} \underline{\ln p} = \underline{X} \underline{\beta}, \quad (3-124)$$

where

$$\underline{X} = \underline{I}_3,$$

the third-order identity matrix, and

$$\underline{\beta}' = (u_1, u_2, u_{12}).$$

Next, finding the estimated asymptotic covariance matrix of $\underline{F(p)}$ from (3-119) or (3-120),

$$\underline{S} = \frac{1}{4} \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\hat{p}_{11}} & 0 & 0 & 0 \\ 0 & \frac{1}{\hat{p}_{12}} & 0 & 0 \\ 0 & 0 & \frac{1}{\hat{p}_{21}} & 0 \\ 0 & 0 & 0 & \frac{1}{\hat{p}_{22}} \end{bmatrix}$$

$$\begin{bmatrix} \hat{p}_{11}(1-\hat{p}_{11}) & -\hat{p}_{11}\hat{p}_{12} & -\hat{p}_{11}\hat{p}_{21} & -\hat{p}_{11}\hat{p}_{22} \\ -\hat{p}_{11}\hat{p}_{12} & \hat{p}_{12}(1-\hat{p}_{12}) & -\hat{p}_{12}\hat{p}_{21} & -\hat{p}_{12}\hat{p}_{22} \\ -\hat{p}_{11}\hat{p}_{21} & -\hat{p}_{12}\hat{p}_{21} & \hat{p}_{21}(1-\hat{p}_{21}) & -\hat{p}_{21}\hat{p}_{22} \\ -\hat{p}_{11}\hat{p}_{22} & -\hat{p}_{12}\hat{p}_{22} & -\hat{p}_{21}\hat{p}_{22} & \hat{p}_{22}(1-\hat{p}_{22}) \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\hat{p}_{11}} & 0 & 0 & 0 \\ 0 & \frac{1}{\hat{p}_{12}} & 0 & 0 \\ 0 & 0 & \frac{1}{\hat{p}_{21}} & 0 \\ 0 & 0 & 0 & \frac{1}{\hat{p}_{22}} \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

(3-125)

$$= \frac{1}{16} \frac{1}{N} \begin{bmatrix} \frac{1}{\hat{p}_{11}} + \frac{1}{\hat{p}_{12}} + \frac{1}{\hat{p}_{21}} + \frac{1}{\hat{p}_{22}} & \frac{1}{\hat{p}_{11}} - \frac{1}{\hat{p}_{12}} - \frac{1}{\hat{p}_{21}} + \frac{1}{\hat{p}_{22}} & \frac{1}{\hat{p}_{11}} - \frac{1}{\hat{p}_{12}} + \frac{1}{\hat{p}_{21}} - \frac{1}{\hat{p}_{22}} \\ \frac{1}{\hat{p}_{11}} - \frac{1}{\hat{p}_{12}} - \frac{1}{\hat{p}_{21}} + \frac{1}{\hat{p}_{22}} & \frac{1}{\hat{p}_{11}} + \frac{1}{\hat{p}_{12}} + \frac{1}{\hat{p}_{21}} + \frac{1}{\hat{p}_{22}} & \frac{1}{\hat{p}_{11}} + \frac{1}{\hat{p}_{12}} - \frac{1}{\hat{p}_{21}} - \frac{1}{\hat{p}_{22}} \\ \frac{1}{\hat{p}_{11}} - \frac{1}{\hat{p}_{12}} + \frac{1}{\hat{p}_{21}} - \frac{1}{\hat{p}_{22}} & \frac{1}{\hat{p}_{11}} + \frac{1}{\hat{p}_{12}} - \frac{1}{\hat{p}_{21}} - \frac{1}{\hat{p}_{22}} & \frac{1}{\hat{p}_{11}} + \frac{1}{\hat{p}_{12}} + \frac{1}{\hat{p}_{21}} + \frac{1}{\hat{p}_{22}} \end{bmatrix}.$$

Since $\hat{p}_{1j} = x_{1j}/N$, (3-125) is identical to (3-102), the asymptotic covariance for (u_1, u_2, u_{12}) . This should be expected since $(\underline{F}(\underline{p}))' = (u_1, u_2, u_{12})$ from (3-124). For this log-linear model the goodness-of-fit test is inappropriate since this model is complete, and from (3-114)

$$\begin{aligned}\underline{b} &= (\underline{X}' \underline{S}^{-1} \underline{X})^{-1} \underline{X}' \underline{S}^{-1} \underline{F} \\ &= (\underline{I}_3 \underline{S}^{-1} \underline{I}_3)^{-1} \underline{I}_3 \underline{S}^{-1} \underline{F} \\ &= \underline{S} \underline{S}^{-1} \underline{F} = \underline{F} = \underline{F}(\hat{\underline{p}}); \end{aligned}$$

so that, from (3-113) with $\underline{X} = \underline{I}_3$,

$$\frac{\chi^2}{F} = (\underline{F} - \underline{X} \underline{b})' \underline{S}^{-1} (\underline{F} - \underline{X} \underline{b}) = 0.$$

In Section 3.3 it was shown that the independence hypothesis test was equivalent to $u_{12} = 0$. In the GSK notation of (3-115)

$$H_0: \underline{K} \underline{\beta} = 0,$$

where

$$\underline{K} = (0, 0, 1).$$

The GSK statistic is then calculated from (3-116),

$$\begin{aligned}
G^2 &= \left[\frac{1}{4} (\ln \hat{p}_{11} - \ln \hat{p}_{12} - \ln \hat{p}_{21} + \ln \hat{p}_{22}) \right]^2 \left[\frac{1}{16} \frac{1}{N} \left(\frac{1}{\hat{p}_{11}} + \frac{1}{\hat{p}_{12}} + \frac{1}{\hat{p}_{21}} + \frac{1}{\hat{p}_{22}} \right) \right]^{-1} \\
&= N (\ln \hat{p}_{11} - \ln \hat{p}_{12} - \ln \hat{p}_{21} + \ln \hat{p}_{22})^2 / \left(\frac{1}{\hat{p}_{11}} + \frac{1}{\hat{p}_{12}} + \frac{1}{\hat{p}_{21}} + \frac{1}{\hat{p}_{22}} \right). \quad (3-126)
\end{aligned}$$

Equation (3-107) showed (3-126) to be an approximation for the MDI statistic, $2I(\underline{m}^*:\underline{x})$, as derived from the Kullback log-linear model. Starting with Kullback's log-linear model, (3-81), and the τ_1 , τ_2 , and τ_3 parameters derived in (3-82), (3-117) can be formed as

$$\underline{F}(\underline{p}) = \underline{C} \ln \underline{p}, \quad (3-127)$$

where

$$\underline{C} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

From (3-112) the model is

$$\underline{F}(\underline{p}) = \underline{X} \underline{\beta}, \quad (3-128)$$

where

$$\underline{X} = \underline{I}_3,$$

and

$$\underline{\beta}' = (\tau_1, \tau_2, \tau_3).$$

The estimated asymptotic covariance matrix, \underline{S} , is then the covariance matrix of the τ parameters, given in (3-101) in terms of the observations, x_{ij} . Writing \underline{S} in terms of the probability estimates, \hat{p}_{ij} ,

$$\underline{S} = \frac{1}{N} \begin{bmatrix} \frac{1}{\hat{p}_{12}} + \frac{1}{\hat{p}_{22}} & \frac{1}{\hat{p}_{22}} & -\frac{1}{\hat{p}_{12}} - \frac{1}{\hat{p}_{22}} \\ \frac{1}{\hat{p}_{22}} & \frac{1}{\hat{p}_{21}} + \frac{1}{\hat{p}_{22}} & -\frac{1}{\hat{p}_{21}} - \frac{1}{\hat{p}_{22}} \\ -\frac{1}{\hat{p}_{12}} - \frac{1}{\hat{p}_{22}} & -\frac{1}{\hat{p}_{21}} - \frac{1}{\hat{p}_{22}} & \frac{1}{\hat{p}_{11}} + \frac{1}{\hat{p}_{12}} + \frac{1}{\hat{p}_{21}} + \frac{1}{\hat{p}_{22}} \end{bmatrix}. \quad (3-129)$$

Knowing that $\tau_3 = 0$ corresponds to the independence hypothesis, (3-115) is again

$$H_0: \underline{K} \underline{\beta} = \underline{0} \quad (3-130)$$

with

$$\underline{K} = (0, 0, 1).$$

The GSK statistic is then calculated from (3-116),

$$\begin{aligned}
G^2 &= [\ln \hat{p}_{11} - \ln \hat{p}_{12} - \ln \hat{p}_{21} + \ln \hat{p}_{22}]^2 \left[\frac{1}{N} \left(\frac{1}{\hat{p}_{11}} + \frac{1}{\hat{p}_{12}} + \frac{1}{\hat{p}_{21}} + \frac{1}{\hat{p}_{22}} \right) \right]^{-1} \\
&= N(\ln \hat{p}_{11} - \ln \hat{p}_{12} - \ln \hat{p}_{21} + \ln \hat{p}_{22})^2 / \left(\frac{1}{\hat{p}_{11}} + \frac{1}{\hat{p}_{12}} + \frac{1}{\hat{p}_{21}} + \frac{1}{\hat{p}_{22}} \right),
\end{aligned}$$

the exact Equation (3-126).

The procedure is easily extended to $2 \times 2 \times 2$ tables and the no second-order interaction hypothesis. Consider Birch's log-linear model, (3-32), and the u parameters in (3-33). For the 2×2 table, only the parameter(s) corresponding to the hypothesis of concern needed to be considered. This is true for any hypothesis testing situation. In (3-117), $\underline{F}(\underline{p}) = \underline{C} \ln \underline{p}$, let

$$\underline{C} = \frac{1}{8}(1, -1, -1, 1, -1, 1, 1, -1)$$

and

$$\begin{aligned}
(\ln \underline{p})' &= (\ln p_{111}, \ln p_{112}, \ln p_{121}, \ln p_{122}, \ln p_{211}, \ln p_{212}, \ln p_{221}, \\
&\quad \ln p_{222}).
\end{aligned}$$

In (3-112), $\underline{F}(\underline{p}) = \underline{X} \underline{\beta}$, let

$$\underline{X} = \underline{I}, \text{ and } \underline{\beta} = (u_{123}).$$

From (3-120)

$$\underline{S} = \frac{1}{64} \frac{1}{N} \left(\frac{1}{\hat{p}_{111}} + \frac{1}{\hat{p}_{112}} + \frac{1}{\hat{p}_{121}} + \frac{1}{\hat{p}_{122}} + \frac{1}{\hat{p}_{211}} + \frac{1}{\hat{p}_{212}} + \frac{1}{\hat{p}_{221}} + \frac{1}{\hat{p}_{222}} \right)$$

Then, for the GSK hypothesis of (3-115), $\underline{K}\underline{\beta} = 0$, let

$$\underline{K} = (1).$$

The GSK statistic from (3-116) is then

$$G^2 = N \frac{(\ln p_{111} - \ln p_{112} - \ln p_{121} + \ln p_{122} - \ln p_{211} + \ln p_{212} + \ln p_{221} - \ln p_{222})^2}{\frac{1}{\hat{p}_{111}} + \frac{1}{\hat{p}_{112}} + \frac{1}{\hat{p}_{121}} + \frac{1}{\hat{p}_{122}} + \frac{1}{\hat{p}_{211}} + \frac{1}{\hat{p}_{212}} + \frac{1}{\hat{p}_{221}} + \frac{1}{\hat{p}_{222}}}$$

(3-131)

CHAPTER IV

EXPERIMENTAL DESIGN

The most important aspect of any study of this nature is the design of the experiment. This design may include several factors, such as the selection of parameters to measure, the selection of independent parameters to control, the settings of these independent parameters, the procedure to use to measure the dependent parameters, and the method to validate the procedure. In making these decisions, careful consideration must be given to the overall purpose of the study and the expected results and possible conclusions. Two other basic considerations, which often come in conflict and often determine the range for the selection of factors above, are the desired level of statistical accuracy and the constraint of cost.

The basic procedure used in this study is a Monte Carlo simulation to approximate exact significance levels for the statistics used in the analysis of contingency tables. With consideration of the above ideas, this chapter discusses the selection of this dependent parameter as well as the selection of the independent parameters, the design of the Monte Carlo simulation, the design of an exact program, and the validation of the Monte Carlo procedure.

4.1 Selection of Parameters

The first step in the design of an experiment is usually to determine the parameters of concern. In this study under the null hypo-

thesis, the dependent or measurable parameter will be the exact level of significance of the test statistics. The independent parameters will be the variables of the underlying sampling distribution. This section will discuss the selection of these parameters, a problem associated with the measure of the dependent parameter, and the selection of the specific values for the independent parameters.

4.1.1 Exact Levels of Significance

As stated in the Introduction, "The primary purpose of this study is to investigate the robustness characteristics of these statistics with respect to small expected values as the size of the table increases." As discussed in Chapter III, the use of these statistics is in hypothesis testing in order to fit an appropriate model to the data and determine the relationships among the variables. In any hypothesis testing situation the procedure is subject to two types of error: type I error, related to the rejection of the null hypothesis when it is true and type II error, related to the acceptance of the null hypothesis when it is false. Usually, the type I error is controlled in that the probability of this error (α) is preselected. In fact, this preselected α , "level of significance", becomes the basis for rejecting or accepting the null hypothesis. The type II error is uncontrolled in the sense that the analyst has little or no knowledge of the actual relationship of the hypothesis. Therefore, the probability of a type II error (β) cannot be calculated specifically. However, in many hypothesis testing situations the test can be designed so that the analyst is "protected", within a designated probability, against making a type II error given a specified alternative hypothesis. This forms the basis for the

operating characteristic curves or power $(1-\beta)$ curves available for most parametric tests.

With respect to contingency table analysis and asymptotic chi-square statistics, as Cochran (1952) states, "there has been little demand for this from applications, because the test is most commonly used when we do not have a clear-cut alternative in mind, and are not in a position to make computations of the power." The work that has been done with respect to type II error or power has been very limited, usually only with 2×2 tables, with assumptions on the underlying distributions, and with results presented only in the asymptotic sense [e.g., Harkness and Katz (1964), Meng and Chapman (1966), and Nathan (1972)]. Under more general alternative hypotheses, the problem with respect to larger contingency tables with small expected values seems intractable.

Reconsidering the type I error, even though it is controlled by the preselection of a nominal level of significance (α) , in the use of these asymptotic chi-square statistics the "exact" level of significance (α_e) is dependent on how well the chi-squared distribution approximates the exact distribution of the statistic. This means that in the use of these statistics there is some "uncontrolled" error with respect to the type I error. This "uncontrolled" error is the difference between the "nominal" and "exact" levels of significance.

Mathematically, the exact level of significance for an approximate chi-square statistic (S) can be calculated as

$$\alpha_e = \sum I_{(S(\underline{x}_1) \chi^2_{\alpha, d.f.})} \cdot f(\underline{x}_1), \quad (4-1)$$

where I is an indicator variable, $S(\underline{x}_i)$ is the value of a statistic given a contingency table realization \underline{x}_i , $f(\underline{x}_i)$ is the probability of obtaining the realization under the sampling model (e.g., multinomial), and $\chi^2_{\alpha, d.f.}$ is the chi-square statistic value at the α nominal level and appropriate degrees of freedom (d.f.). The sum is taken over all possible table realizations given the sample size N for the multinomial sampling model.

It should be apparent that these exact levels will generally be different than the nominal levels. The chi-squared distribution is continuous, so that any α , $0 < \alpha < 1$, could be chosen. Yet, for a given sample size there are only a discrete number of possible contingency table arrangements, so only a discrete number of possible α_e .

α_e is the "exact" probability of rejecting a true hypothesis. Intuitively, in order for the test statistic to perform well, these exact levels should be close to the nominal levels - the closer, the better. If the exact level is less than the nominal level, the test is conservative, and if the exact level is greater than the nominal level, the test is liberal. A conservative test is usually preferred since the nominal level provides a bound on the probability of rejecting a true hypothesis. Yet, in general, β increases as α decreases, so that a liberal test may be preferred in some situations to protect against a type II error. This may be particularly important in independence hypothesis testing situations, where the purpose often is to find variables that are independent (i.e., accept the null hypothesis).

The regression example in Chapter I demonstrates the problems that can occur when the chi-squared distribution does not well approxi-

mate the exact distribution of these statistics. The exact levels of significance provide a good parameter for comparing the performance of these statistics when used in hypothesis testing. From the discussion above, both liberal and conservative situations need to be considered. The best criterion would seem to be how close these exact levels are to the nominal levels. In this study this criterion will provide the basis for the comparison of these statistics.

4.1.2 Zero Cells and Zero Marginals

The presence of cells containing zero entries and of marginals having zero values can cause significant problems in the analysis of contingency tables and, in particular, in the use of the chi-square statistics. Zero entries in contingency tables are primarily of two types: structural or sampling. Structural zeros are the result of sampling variables where some cross-classification cannot physically occur. An example would be in the classification of male and female surgical operations where there could be no cross-classified cell for male hysterectomies. Tables of this nature are generally defined as "incomplete" tables, and specialized procedures are available to handle them [e.g., see Bishop and Fienberg (1969), Mantel (1970), or Chen and Fienberg (1976)]. These situations will not be investigated in this study. Sampling zeros, on the other hand, are the result of the fact that there is some probability of obtaining an observed zero entry in a table of finite sample size under a given sampling model, such as the multinomial. But, how do these sampling zeros affect the analysis of contingency tables using the methodologies considered in this study?

Both the Pearson X^2 and Kullback 2I statistics depend on the

calculation of maximum likelihood estimates (MLE). For the Pearson X^2 , as long as the MLE are non-zero, the statistic can be calculated even in the presence of zero observed cells. The Kullback statistic ($2I = 2 \sum_i x_i \ln(x_i/\hat{m}_i)$) depends, not only on non-zero MLE, but also on non-zero cell entries. But, noting that $\lim_{n \rightarrow \infty} n \ln n = 0$, the contribution to the value of the Kullback statistic for cells with zero observed entries is essentially zero. In practice, the value of this contribution is set at zero, or as suggested by Gokhale and Kullback (1978b), some small value (e.g., .000001) is given to the zero cell. This procedure seems to cause very little bias.

The GSK statistic, G^2 , is the most affected by the presence of zero observed cells. For example, consider the 2×2 independence test statistic,

$$G^2 = (\ln x_{11} - \ln x_{12} - \ln x_{21} + \ln x_{22})^2 / \sum_i \sum_j (1/x_{ij}).$$

When any observation is zero, both the numerator and demoninator become infinite. There is no easy way to separate the contribution of a single cell, such as with the Pearson X^2 and Kullback $2I$, and show a limiting process. Several authors have suggested ways to overcome this problem [e.g., see Grizzle, Starmer, and Koch (1969), Goodman (1970), Fienberg and Holland (1970), and Bhapkar (1979)]. Grizzle et.al. (1969) recommend adding $1/r$ to each zero cell, where r is the number of response categories (with multinomial sampling, r is the number of cells). This "1/r rule", an extension of a rule given by Berkson (1955) for binomial sampling, is the most popular rule in use today and will

be used in this study. As Forthofer and Lehnen (1981, p. 14) suggest, "Replacement of the zero frequency with a small, non-zero value introduces a slight but acceptable bias into the estimate."

A more significant problem in the use of these statistics occurs when a marginal has an observed zero value. With respect to the Pearson and Kullback statistics, the problem is easily seen since these statistics depend on calculations of MLE. As shown in Chapter III, the MLE depend on the marginals corresponding to the hypothesis of concern. If an observed marginal is zero, meaning that all the corresponding cells for that marginal are zero, then the MLE for those cells will be zero since these marginals must be preserved. Neither the Pearson nor Kullback statistics can be calculated with zero MLE.

Under the presense of zero marginals corresponding to the hypotheses of concern, a more fundamental problem occurs affecting all three statistics. This problem involves the very nature of the hypotheses and is best described through the log-linear model.

Consider the 2×2 table and the log-linear model reparameterized in (3-15). The hypothesis of independence corresponds to $u_{12} = 0$ (or τ_3 of Kullback's log-linear model). Assume that one of the marginals is equal to zero, say $x_{.1} = 0$, and the others are non-zero as reflected in Figure 9.

0	x_{12}	$x_{1.}$
0	x_{22}	$x_{2.}$
0	$x_{.2}$	

Figure 9. 2×2 Table: One Zero Marginal ($x_{.1} = 0$)

Under the null independence hypothesis the MLE can be calculated from (3-52),

$$\hat{m}_{11} = (x_{1.})(0) = 0$$

$$\hat{m}_{12} = (x_{2.})(x_{.2})$$

$$\hat{m}_{21} = (x_{2.})(0) = 0$$

$$\hat{m}_{22} = (x_{2.})(x_{.2})$$

The MLE provide estimates for the probabilities,

$$\hat{p}_{ij} = \hat{m}_{ij}/N.$$

These estimates for p_{11} and p_{21} are zero, and from (3-19) u_{12} (or (3-82) τ_3) cannot be estimated. In fact, there is not enough information to perform the independence test under any criteria.

Consider now the 2×3 table with one of the three category marginals equal to zero, say $x_{.1} = 0$, as reflected in Figure 10.

0	x_{12}	x_{13}	$x_{1.}$
0	x_{22}	x_{23}	$x_{2.}$
0	$x_{.2}$	$x_{.3}$	

Figure 10. 2×3 Table: One Zero Marginal ($x_{.1} = 0$)

The corresponding MLE, \hat{m}_{11} and \hat{m}_{21} , are equal to zero. In the 2×3

log-linear model there are two interaction terms. The two degrees of freedom independence test corresponds to both terms being equal to zero. Depending on the parameterization used, these interaction terms can have different characterizations. For Kullback's log-linear model these terms are

$$\tau_3 = \ln p_{11} - \ln p_{21} + \ln p_{23} - \ln p_{13}$$

$$\tau'_3 = \ln p_{12} - \ln p_{22} + \ln p_{23} - \ln p_{13}.$$

Since $\hat{m}_{11} = \hat{m}_{21} = 0$, there is no estimate for τ_3 , and the two degree of freedom test cannot be performed. However, a one degree of freedom test could be performed based only on τ'_3 . In effect, this would be collapsing the table to a 2×2 table, disregarding the zero cells of the zero marginal. The τ'_3 term would correspond exactly to the τ_3 term of the 2×2 Kullback log-linear model. The only effect of the zero marginal would be that a degree of freedom is lost.

In general, this problem would extend to larger contingency tables and other hypotheses. A zero marginal, where the marginal relates to the hypothesis of concern, will cause a loss in the degrees of freedom of the test and, in effect, collapse the table. It should be noted that this concept of collapsing differs from the usual concept of collapsibility of contingency tables as described by Bishop (1971) and Bishop et.al. (1975, p. 47).

Because of this problem associated with zero marginals, the exact

(or estimated) levels of significance in this study will be calculated conditionally on these marginals, associated with the hypothesis of concern, not being equal to zero. These levels of significance will then be conditional probabilities. From the discussion above, this is intuitively desirable. When marginals are zero, degrees of freedom are lost, and the hypothesis test is made with the appropriate lower degree of freedom chi-squared distribution. The collapsed contingency table will, in effect, be an observation already considered with respect to the calculation of exact levels of significance for these smaller tables.

4.1.3 Probability Designs

The exact levels of significance are calculated from (4-1), based not only on the null hypothesis and the statistic but also the underlying sampling model. For the tests of this study the sampling model is multinomial with parameters of sample size and cell probabilities. These provide the natural independent parameters for the experiment. The selection of sample sizes will be discussed in the next section. This section will discuss the selection of the cell probabilities, which in vector form will be probability vectors.

In selecting these probability vectors, two considerations are important. First, the design must be large enough in terms of the range of probabilities to encompass the expected range of variability of the exact levels of significance and provide trend information. Second, the design must be small enough in terms of number of vectors selected to be manageable within the constraint of available computer time.

The null hypotheses of independence conveniently provide the basis for the selection of the underlying probability vectors. For example, the null hypothesis of independence for two-way tables is

$$H_0: p_{ij} = p_{i.}p_{.j}; i=1,2,\dots,r; j=1,2,\dots,s.$$

By specifying the marginal probabilities ($p_{i.}, p_{.j}$), under the null hypothesis the complete probability structure is determined. A similar situation exists for three-way tables and the complete independence hypothesis, where the one-way marginals ($p_{i...}, p_{.j...}, p_{...k}$) are sufficient.

Table 5 provides a partitioning of the degrees of freedom associated with each table and hypothesis test used in this study. Included in the table is a list of those marginals to be specified. Note that not all marginal probabilities need be specified since the multinomial constraint, that the probabilities sum to one, will uniquely determine one marginal if the other corresponding marginals are given. In Chapter III it was shown that the hypothesis, $H_0: p_{11} = p_{1.}p_{.1}$, for the 2×2 table was sufficient for the one degree of freedom hypothesis test for independence (which requires that $p_{ij} = p_{i.}p_{.j}$ for all i, j). Similar results hold for other tables and independence tests, and the required hypothesis tests for this study are listed in Table 5. Other combinations of tests could be chosen that would also suffice. The ones listed in Table 5 correspond to the lowest lexicographic order.

Except for the no second-order interaction test in the $2 \times 2 \times 2$ table, the tests and required marginals in Table 5 are fairly straightforward. For the no second-order interaction test a simple combination

Table 5. Hypothesis Tests and Specified Marginals

Table	Degrees of Freedom*	Hypothesis Tests and Specified Marginals
2×2	1	Independence H_0 : $p_{11} = p_{1.}p_{.1}$
	2	Specify $p_{1.}, p_{.1}$
2×3	2	Independence H_0 : $p_{11} = p_{1.}p_{.1}, p_{12} = p_{1.}p_{.2},$
	3	Specify $p_{1.}, p_{.1}, p_{.2}$
2×4	3	Independence H_0 : $p_{11} = p_{1.}p_{.1}, p_{12} = p_{1.}p_{.2},$ $p_{13} = p_{1.}p_{.3}$
	4	Specify $p_{1.}, p_{.1}, p_{.2}, p_{.3}$
3×3	4	Independence H_0 : $p_{11} = p_{1.}p_{.1}, p_{12} = p_{1.}p_{.2},$ $p_{21} = p_{2.}p_{.1}, p_{22} = p_{2.}p_{.2}$
	4	Specify $p_{1.}, p_{2.}, p_{.1}, p_{.2}$
2×5	4	Independence H_0 : $p_{11} = p_{1.}p_{.1}, p_{12} = p_{1.}p_{.2},$ $p_{13} = p_{1.}p_{.3}, p_{14} = p_{1.}p_{.4}$
	5	Specify $p_{1.}, p_{.1}, p_{.2}, p_{.3}, p_{.4}$

*Each table has 1 degree of freedom associated with the normal constraint $\sum p = 1$.

Table 5. Continued

Table	Degrees of Freedom*	Hypothesis Tests and Specified Marginals
$2 \times 2 \times 2$	4	Complete Independence $H_0: p_{111} = p_{1..}p_{.1.}p_{..1},$ $p_{112} = p_{1..}p_{.1.}p_{..2},$ $p_{121} = p_{1..}p_{.2.}p_{..1},$ $p_{211} = p_{2..}p_{.1.}p_{..1}$
	3	Specify $p_{1..}, p_{.1.}, p_{..1}$
$2 \times 2 \times 2$	7	Complete Independence $H_0: p_{111} = p_{1..}p_{.1.}p_{..1},$ $p_{112} = p_{1..}p_{.1.}p_{..2},$ $p_{113} = p_{1..}p_{.1.}p_{..3},$ $p_{121} = p_{1..}p_{.2.}p_{..1},$ $p_{122} = p_{1..}p_{.2.}p_{..2},$ $p_{211} = p_{2..}p_{.1.}p_{..1},$ $p_{212} = p_{2..}p_{.1.}p_{..2}$
	4	Specify $p_{1..}, p_{.1.}, p_{..1}, p_{..2}$
$2 \times 2 \times 2$	1	No Second-Order Interaction $H_0: p_{111}p_{122}p_{212}p_{221} =$ $p_{112}p_{121}p_{211}p_{222}$
	6	Specify $p_{11.}, p_{12.}, p_{1.1}, p_{2.1}, p_{.11}, p_{.12}$

*Each table has 1 degree of freedom associated with the normal constraint $\sum p = 1$.

of one-way marginals will not suffice. The hypothesis has only one degree of freedom, leaving six degrees of freedom for the specification of parameters to determine uniquely the p_{ijk} under the null hypothesis. The combination of given two-way marginals will suffice, as well as any other independent set. With these two-way marginals an iterative procedure, similar to that given for the MLE in Chapter III, could be used to determine uniquely the cell probabilities. An easier combination of six independent parameters would be any six p_{ijk} . The other two p_{ijk} could then be computed from the null hypothesis and the multinomial constraint, $\sum_{i,j,k} p_{ijk} = 1$.

Once the probability parameters have been designated, the question still remains of how these parameters should be varied in order to insure a sufficient range for the probability structure. After some preliminary investigation and consideration of the need to have some structures with very small expected values, it was decided to allow the marginals to have values as small as .1. In effect, this would provide some cells with probabilities as small as .01 for the two-way tables and .001 for the three-way tables. Using sample sizes under 100 this would guarantee some tables with very small expected values.

Considering first the 2×2 table, by allowing the marginals $(p_{1.}, p_{.1})$ to range from .1 to .9 in increments of .1 and taking the complete convolution, there would be 81 vectors. However, these vectors are not unique. It makes no difference which variable is variable one and which variable is variable two. The interaction term of the log-linear model, u_{12} , will be unaffected; so that, letting $p_{1.} = .1$ and $p_{.1} = .2$ is equivalent to letting $p_{1.} = .2$ and $p_{.1} = .1$. This reduces

the number of vectors to 45.

Additionally, there is a symmetry with respect to the categories of each variable. A simple proof will demonstrate this symmetry. Let $p_{1.}$ and $p_{.1}$ be selected, then under H_0

$$p_{11} = p_{1.}p_{.1}$$

$$p_{12} = p_{1.}p_{.2}$$

$$p_{21} = p_{2.}p_{.1}$$

$$p_{22} = p_{2.}p_{.2},$$

where the probabilities can be represented in the standard arrangement of Figure 11.

p_{11}	p_{12}	$p_{1.}$
p_{21}	p_{22}	$p_{2.}$
$p_{.1}$	$p_{.2}$	1

Figure 11. 2×2 Table: Standard Arrangement

Now, let another selection of the marginals be made where $p'_{1.} = p_{1.}$

and $p'_{.1} = 1 - p_{.1} = p_{.2}$, then under H_0

$$p'_{11} = p'_1 \cdot p'_{.1} = p_1 \cdot p_{.2} = p_{12}$$

$$p'_{12} = p'_1 \cdot p'_{.2} = p_1 \cdot p_{.1} = p_{11}$$

$$p'_{21} = p'_2 \cdot p'_{.1} = p_2 \cdot p_{.2} = p_{22}$$

$$p'_{22} = p'_2 \cdot p'_{.2} = p_2 \cdot p_{.1} = p_{21}$$

with the arrangement of original probabilities reflected in Figure 12.

p_{12}	p_{11}	$p_{1.}$
p_{22}	p_{21}	$p_{2.}$
$p_{.2}$	$p_{.1}$	1

Figure 12. 2×2 Table: Interchanged Categories

In effect, the categories of variable two have been interchanged. With respect to the log-linear model this change will simply change the sign of the interaction term. The chi-square tests for independence will be unchanged.

These symmetry arguments have now reduced the 81 vectors to only 15. The marginals, $p_{1.}$ and $p_{.1}$, range from .1 to .5 by increments of .1 and are convoluted to form the complete design (given in Appendix B) for the 2×2 table.

Symmetry arguments for the 2×3 table are similar with respect to change of categories, but the variables cannot be interchanged since they each have a different number of categories. However, results for the 3×2 table would be equivalent to those for the 2×3 table. From

Table 5 there is a requirement to specify the three marginals ($p_{1.}, p_{.1}, p_{.2}$). Letting $p_{1.} = .1$ and range to $.5$, $p_{.1} = .1$ and range to $.3$, and $p_{.2} = p_{.1}$ and range to $.4$, all in increments of $.1$, the entire convolution will include 40 vectors. From a preliminary investigation it was thought that it would be too costly in terms of computer time to use the entire design, and that similar information could probably be obtained using only one-half the number of vectors. It was decided to use every other vector in the lexicographic ordered design of varying marginals. In addition, the equiprobable vector with marginals $p_{1.} = 1/2$ and $p_{.1} = p_{.2} = 1/3$, giving equiprobable cell probabilities $p_{ij} = 1/6$ for all i and j , was included. The final design of 21 vectors is given in Appendix B.

The 2×4 and 2×5 tables had similar complete designs of 40 and 32 vectors, respectively. Again, every other vector was used, and the equiprobable case added to the 2×4 table, to give designs with 21 and 16 vectors. The 3×3 table, like the 2×2 table, has the additional symmetry property of interchange of variables. A total of 36 vectors were available in the complete design. Including the equiprobable case, 21 vectors were used for the study. All these designs are listed in Appendix B.

The $2 \times 2 \times 2$ table under the hypothesis of complete independence, like the 2×2 and 2×3 tables, has the symmetry property of interchange of any two variables, as well as the interchange of categories, without affecting the chi-square test statistics. The complete design includes 34 vectors. Under a similar scheme of roughly including every other vector and the equiprobable vector, 22 vectors were used for the design.

The $2 \times 2 \times 2$ table under the hypothesis of no second-order interaction presents some specialized problems. First, as reflected in Table 5, under this hypothesis there is no convenient set of marginals that will easily generate the cell probabilities. If marginal probabilities, such as those in Table 5, are used, an iterative scheme would be needed to find the cell probabilities. Any six cell probabilities could be specified to satisfy the six degrees of freedom, but any attempt to cover the range of possibilities for these specified probabilities would seem fruitless. As noted in Chapter III, the no second-order interaction hypothesis is a sub-hypothesis of the complete interaction hypothesis; so that, the probability vectors selected under complete interaction also satisfy the hypothesis of no second-order interaction. Therefore, it was decided to use this *same design* for no second-order interaction hypothesis in the $2 \times 2 \times 2$ table. Unfortunately, another problem occurred. As discussed in Section 4.1.2, zero observed marginals corresponding to the hypothesis of concern will invalidate the test at the given degrees of freedom. The twelve two-way marginals of the $2 \times 2 \times 2$ table correspond to the no second-order interaction hypothesis. In the generation of random tables, restricting these marginals to be non-zero for the extreme vectors in the $2 \times 2 \times 2$ complete independence design causes many observations to be discarded. A preliminary study, using the most extreme probability vector, required about 800 tables to be generated in order to find five that had no two-way marginals equal to zero. It was decided to discard the five most extreme vectors, leaving the design of 17 vectors listed in Appendix B.

For the $2 \times 2 \times 3$ table, under the complete independence hypothesis the number of vectors for the complete design is increased to 120. This is the result of the fact that the third variable cannot be interchanged with the other two; hence, a complete convolution with respect to this variable is required. To reduce the number of vectors used and still cover the range of possible vectors, it was decided to use every fourth vector and include the equiprobable vector to give a design of 31 vectors as listed in Appendix B.

This completes the designs of the underlying probability structures for the various tables and hypothesis tests. It is believed that these reflect the range of what would occur in actual hypothesis testing situations and should give indications of trends with respect to any parameters that may be considered.

4.1.4 Sample Sizes

While specific sample sizes may not be as important as the underlying probability structure, care should be taken to allow consideration of those sample sizes, given the probability structures, that will allow cell expected values to range from relatively small to relatively large. The relationship between sample size (N) and expected value (m) is $m = Np$, where p is the probability of the corresponding cell.

It might be expected that sample sizes can be as small as desired. Unfortunately, the zero-marginal constraint, previously discussed, reasonably requires that some lower bound be placed on the sample size. The one exception to this is the 2×2 table where an "exact" procedure and program were developed, making it very easy to calculate exact levels of significance for any small sample size (N a multiple of four).

The other tables use a Monte Carlo procedure where small sample sizes require many more tables to be generated before a sufficient number of "usable" (no zero marginals corresponding to the hypothesis of concern) tables are available.

For a maximum sample size value, somewhere near 100 was thought to be sufficient. This would give expected values for the equiprobable tables from 10 for the 2×5 table to 25 for the 2×2 table and from 8.33 for the $2 \times 2 \times 3$ table to 12.5 for the $2 \times 2 \times 2$ table. Based on a preliminary investigation, 100 also seemed reasonable with respect to computer generation times.

Having chosen the range of values, it was important to consider what specific sample sizes would be used for each design. It was thought that, since comparisons were to be made across tables, some criteria to standardize these sample sizes would be useful. Consider the parameter, average expected value, $\bar{m} = N/k$, where the k is the number of cells ($k=rs$ for a two-way table). Roscoe and Byars (1971) based their recommendations on this quantity. Usually, the magnitudes of the expected values (based on estimates) are unknown until after a sample is taken. Because of this, Roscoe and Byars felt that their results would be most valuable reported for their uniform (equiprobable) case. Also, they state that, "there is an appealing simplicity to this approach."

What seems even more important for the study of this thesis is the need for comparisons across tables. With this in mind sample sizes were chosen so that the average expected values would be integer. The general comparable range across tables for these average expected values are from two to 12, although the 2×2 and 2×3 tables, in particular,

include higher values. Also, during the testing of each Monte Carlo program, some adjustments of sample sizes were made to control computer times. For example, the minimum N for the $2 \times 2 \times 2$ table under the no second-order interaction hypothesis was changed from 16 to 24 due to previously discussed zero-marginal problems.

The complete sample size design is given in Appendix C. These sample sizes were used with each probability vector of the probability designs in Appendix B.

4.2 The Exact 2×2 Program

The primary purpose of developing an exact program for the 2×2 table was to provide data for validating the Monte Carlo simulation. The exact data, along with Monte Carlo data for large sample sizes, were also used in the basic analysis of the 2×2 table. A listing of the exact program is given in Appendix D, and the exact data generated for the complete 2×2 design are given in Appendix F.

The procedure used in the program is based on a partial enumeration of all possible tables and on symmetry properties. The key to the procedure is restricting the sample size N to be divisible by four. This allows for significant computational efficiencies to be achieved.

Tables were first classified based on their symmetry properties with respect to the three chi-square statistics. Formulas were then derived which provided information on the number of tables to be generated for a given sample size. Then, procedures were developed to enumerate the tables within each classification.

The first formula derived was for the total number of "usable"

tables for a given sample size. This formula for any $N > 1$ is

$$(N^3 + 6N^2 - 3N + 6)/6.$$

However, many of these tables are not unique in the sense that, due to symmetry properties, a chi-square statistic will have the same value for several tables. Formulas for the number of "unique" (with respect to the three statistics), "usable" tables were derived for various sample sizes. These are

$$(N^3 + 9N^2 - N - 9)/48 \quad \text{for } N \text{ odd,}$$

$$(N^3 + 9N^2 + 8N)/48 \quad \text{for } N \text{ even and divisible by 4,}$$

$$(N^3 + 9N^2 + 8N - 12)/48 \quad \text{for } N \text{ even and not divisible by 4.}$$

Table 6 gives a comparison of the total and unique number of tables for various sample sizes. From the formulas it is easy to see that the ratio

Table 6. 2×2 Total and Unique Number of Tables

<u>N</u>	<u>Total</u>	<u>Unique</u>	<u>N</u>	<u>Total</u>	<u>Unique</u>	<u>N</u>	<u>Total</u>	<u>Unique</u>
4	19	5	12	407	65	32	6417	880
5	36	7	14	624	96	36	8995	1221
6	60	12	16	905	136	40	12181	1640
7	92	16	18	1258	185	44	16039	2145
8	133	24	20	1691	245	48	20633	2744
9	184	30	24	2829	400	52	26027	3445
10	246	41	28	4383	609	56	32285	4256

of total to unique tables is approaching $1/8$. Using the symmetry properties, at $N = 56$ only 13 percent of the total number of tables need to be generated - a considerable savings.

Considering the symmetry properties, the tables were classified into five groups (E_1, E_2, E_3, E_4, O). Table 7 presents this classification scheme for N divisible by four along with formulas for the number of unique tables, formulas for the number of total tables within each classification, and a specification of the number of arrangements for each unique table within each classification group. The primary purpose for this classification scheme relates to how many of the total number of tables are equivalent to a given "unique" table; i.e., how many symmetric arrangements are possible for each unique table. For example, group E_1 defines those tables with all cells equal, but for a given sample size (N) divisible by four, there is only one such table. Group E_2 defines those tables where both sets of corners are equal (in lexicographic order this is (abab) with $a \neq b$). There are $N/4$ such "unique" tables, and for each "unique" table there are two arrangements included in the total number of tables; i.e., (abab) and (baba). Note that the total number of tables can be found by multiplying the number of unique tables by the number of arrangements, and that the grand totals correspond to the formulas previously given.

The exact levels of significance are essentially calculated by summing multinomial probabilities for tables where the particular approximate chi-square statistic is greater than the value of the chi-square statistic (with one degree of freedom) at a specified nominal level. The procedure begins by generating a table, $\{x_{ij}\}$, within a

Table 7. 2×2 Symmetric Classifications (N divisible by 4)

Label	Classification	No. Unique	No. Arrangments/Unique	Total No.
E_1	All Cells Equal (aaaa)	1	1	1
E_2	Both Corners Equal (abab) $a \neq b$	$N/4$	2	$N/2$
E_3	Both Sides Equal (aabb) $a \neq b$	$(N-4)/4$	4	$N-4$
E_4	One Set Equal Corners (abac) $b \neq c$	$\frac{N^2+2N-8}{8}$	4	$\frac{N^2+2N-8}{2}$
0	All Others	$\frac{N^3+3N^2-28N+48}{48}$	8	$\frac{N^3+3N^2-28N+48}{6}$
<hr/>				
Totals		$\frac{N^3+9N^2+8N}{48}$		$\frac{N^3+6N^2-13N+6}{6}$

particular symmetric classification. The total multinomial probability,

$$f(x_g) = \sum_{x_{ij} \in x_g} f(x_{ij}) = \sum_{x_{ij} \in x_g} N! \prod_{i,j} p_{ij}^{x_{ij}} / x_{ij}!,$$

is calculated for each unique table group (x_g) by using the generated table and rotating it through its symmetric arrangements. The three statistics, S_{ig} ($i=1,2,3$), are then calculated for this particular unique table group. Hypothesis tests are then performed at each nominal level of significance ($\alpha = .10, .05, .01$) by comparing the statistics with the $\chi^2_{\alpha,1}$ statistics. The multinomial probability associated with a rejected hypothesis is accumulated,

$$\sum_{\text{all } g} f(x_g) \cdot I_{ig}^{\alpha},$$

where I_{ig}^{α} is an indicator variable for the hypothesis test; i.e.,

$$I_{ig}^{\alpha} = \begin{cases} 1, & \text{if } S_{ig} > \chi^2_{\alpha,1} \\ 0, & \text{otherwise} \end{cases}.$$

Also, the total used probability is accumulated,

$$F = \sum_{\text{all } g} f(x_g).$$

The sums are taken over all unique table groups. It should be noted that F will not equal one since only "usable" tables are considered.

Exact significance levels for each statistic (S_i) and each nominal α are then calculated as the ratio of these two accumulations,

$$A_{S_i} = \sum_{\text{all } g} (f(x_g) \cdot I_{ig}^\alpha) / F.$$

It is clear that these exact significance levels are conditional probabilities.

As a check that all required tables were considered, an exact formula for F was derived. Letting Z be the probability of a zero marginal,

$$Z = p_{1.}^N + p_{2.}^N + p_{.1}^N + p_{.2}^N - \sum_{i,j} p_{ij}^N.$$

F is the probability of a "usable" table, so

$$F = 1 - Z.$$

F values were calculated for each design point and compared to the accumulated totals in the program. Agreement was exact to the four decimal places of the output data.

4.3 The Simulation Program

The Monte Carlo simulation program is the heart of this study. It provides most of the data for the analysis and the results of Chapter V. The purpose of the simulation is to provide estimates of exact levels of significance for the competing statistics at designated

nominal levels and under the probability and sample size designs presented in Section 4.1. This section will discuss the design of the Monte Carlo program and its validation and the selection of the number of tables to generate for each design point.

4.3.1 Program Design

A separate computer program was written for each type table (2×2 , 2×3 , ..., $2 \times 2 \times 3$) and hypothesis test (independence, complete independence, no second-order interaction). The efficiency of the program was paramount, and by using separate programs, maximum advantage could be taken of the unique structure of each table and hypothesis test. Appendix E provides a listing of the program for the 2×3 table. Each of the other programs follows the same general procedure and formats.

Each program is divided into four parts: a main program and three subroutines. The main program controls the cycling process through the various probability designs and sample sizes, calls the three subroutines, makes the final calculations for the estimated exact levels of significance, and prints the results. The three subroutines will be discussed below in the order that they are used.

Subroutine GEN is the most important part of the simulation. The subroutine uses the IMSL (1980) subroutine GGMTN coding, modified, for efficiency, to the specific problem of generating random multinomial observations of size k (k equal to the number of cells) with parameters given by the probability/sample size designs. The procedure begins by generating a binomial random observation based on using the multinomial probability (p_1) for a cell and the sample size, N . The binomial observation will be this cell's multinomial observation

(x_1) . To generate the observation for the next cell, its multinomial probability is first adjusted to a conditional probability (given that the first cell has been observed) by dividing by the unused probability ($p'_2 = p_2 / (1 - p_1)$). The sample size is also adjusted by subtracting the random observation of the first cell ($N' = N - x_1$). These parameters (N' , p'_2) are then used to generate another random binomial observation, which is the multinomial observation (x_2) for the second cell. The procedure is continued until one cell remains. The observation for this cell is, with conditional probability equal to one, equal to the unused sample size ($N - x_1 - x_2 \dots - x_{k-1}$).

The procedure above includes the use of the IMSL binomial random variate generator, subroutine GGBN. This subroutine uses a basic counting routine for N less than 35, where uniform $(0,1)$ random numbers are generated and compared to the binomial probability p , then p^2 , p^3 , and p^4 . This procedure allows for up to four realizations of the binomial random variate with only a generation of one uniform random number. For N greater than 34 GGBN uses a method reported by Relles (1972) based on the use of medians from random samples of a uniform distribution. The medians are "efficiently" generated as beta random variates.

The procedure above for N less than 35 is exact, in the sense that the "randomness" of the observations depends only on the random number generator itself. Relles' procedure for N greater than 34 is approximate. Relles derives upper bounds on the probability of error in using his procedure for various "threshold" values of N . In the GGBN subroutine the threshold value is 34. For a threshold value of

31 Relles reports error probabilities of .00030, .00042, and .00046 for sample sizes of 32, 64, and 128, respectively. For the study of this thesis a reasonable bound for the probability of an error in any one cell would then be .0005. An error, once made, would propagate through the other cells, but the probability of making two errors in one multinomial observation would be bounded by $(.0005)^2$. This bound for the probability of multiple errors would continue to multiply until the remaining N' was less than 35 and the "exact" counting method used. It is clear that this procedure will be quite accurate.

The IMSL routines for random number generation used in the GGBN subroutine are GGUBS and GGUFBS, where GGUFBS is the function form of GGUBS. These routines use a linear congruential generator first reported by Hutchinson (1966). The particular version with multiplier 16807 and modulus $2^{31}-1$ was first presented and tested by Lewis, Goodman, and Miller (1969) for the IBM 360 computer. The generator has been extensively tested by Learmonth and Lewis (1973) and shown to have excellent statistical properties.

As part of subroutine GEN, after a multinomial random observation is generated, the corresponding contingency table is checked for appropriate zero marginals. If a marginal is zero, the observation is discarded and a new observation is generated. The process continues until an appropriate number of "usable" random tables have been generated.

Once these tables have been generated, subroutine STATS is called. This subroutine calculates maximum likelihood estimates and the three contingency table statistics for each table. The calculations for the Pearson and Kullback statistics are fairly straightforward. Each cell

provides a contribution to the value of the statistic, with zero cells providing a contribution of zero to the Kullback statistic (see Section 4.1.2). The calculation of the GSK statistic is more involved. As presented in Chapter III, this statistic requires an inversion of the asymptotic covariance matrix. Conveniently, for the $2 \times s$ tables this matrix is highly structured with all off-diagonal terms equal, and the inversion can be derived in a relatively simple closed-form. For the $2 \times 2 \times 2$ table under the complete independence null hypothesis, the four degree of freedom GSK test statistic requires the inversion of a 4×4 matrix that does not have the convenient structure of the $2 \times s$ tables. An efficient IMSL symmetric matrix inversion subroutine, L1NV1P, was used. As will be discussed in Chapter V, the GSK statistic was not calculated for the 2×5 , 3×3 and $2 \times 2 \times 3$ tables.

Once the statistics for each table are calculated, subroutine HYP is called. This subroutine performs a hypothesis test on each of the tables for each statistic at each nominal level of significance (.10, .05, .01), based on the value of the chi-square statistic specified to six significant digits [Pearson and Hartley (1954, Vol. I, Table 8, pp. 130-134)]. The subroutine accumulates the number of times the hypothesis test rejects for each statistic at each nominal level.

Once all of the tables have been tested, the rejection totals are returned to the main program. The main program calculates the estimated "exact" levels of significance by dividing the rejection totals by the total number of tables generated. These estimates are then printed for each probability/sample size design, statistic, and nominal level.

4.3.2 Number of Tables

A critical aspect in the design of the Monte Carlo procedure was the determination of the number of tables to generate for each estimate of the exact levels of significance. The previous studies, reviewed in Chapter II, used 1000, 2000, 5000, and 10000 tables. With the size of the designs to be considered in this study, it seemed crucial to limit the number of tables to be as low as possible and still give statistically reliable data. It was decided to consider accuracy levels, corresponding to confidence intervals about the nominal levels of significance, for various numbers of tables. These levels depend on the normal approximation to the binomial distribution.

From the procedure described in the previous section, let X be the number of "rejections" in K tables. Then X has a binomial distribution with probability p , where p is the "exact" probability of rejection, and the mean and variance are $\mu = Kp$ and $\sigma^2 = Kp(1-p)$. Then $(X-Kp)/(Kp(1-p))^{1/2}$ has an approximate normal $(0,1)$ distribution for large K . Letting E be the maximum "error" (accuracy) for the random variable X , with probability $1-\alpha$, $E = Z_{\alpha/2}(p(1-p)/K)^{1/2}$. For the nominal values of p (.10, .05, .01) to be considered in this study, errors were calculated with probability .95 (corresponding to 95 percent confidence intervals) for various numbers of tables (K) and are shown in Table 8.

Besides these accuracy levels, preliminary computer program tests were performed to estimate computer times and to observe the effects of increasing K by using a moving average of estimated levels. Based on these tests, the previous studies, and the estimated .95 accuracy levels of Table 8, 2000 tables were selected. A significant factor in

Table 8. .95 Accuracy Levels (E)

<u>Nominal p</u>	<u>K</u>	<u>E</u>
.10	1000	.0186
	2000	.0131
	5000	.0083
	10000	.0059
.05	1000	.0135
	2000	.0096
	5000	.0060
	10000	.0043
.01	1000	.0062
	2000	.0044
	5000	.0028
	10000	.0020

K = No. Tables

this selection was the .95 accuracy level at the nominal value of .05. Table 8 shows this level to be .0096, which would correspond to a 95 percent confidence interval approximately (.04, .06). Several authors, beginning with Cochran (1952), have used this interval as a measure of performance of the chi-square statistics at the .05 nominal level.

4.3.3 Program Validation

In order for any simulation model to be useful as an investigative tool, it must be reliable and consistent; that is, the simulation should provide reasonable results. To check these properties, it is often desirable to provide a method of validation. Validation is usually accomplished by checking the output of the simulation for some of the values of the input parameters where the true output is known. This is especially important with discrete event simulation. In

Monte Carlo simulation this is often not essential since exact formulas are usually used for the generation procedure, and empirical results or tests are available to provide sufficient information on the simulation validity. This is the case for the Monte Carlo simulation of the study in this thesis. Errors could occur as a result of problems with the random number generator or as a result of the approximate binomial generation scheme for N greater than 34. As previously discussed, the tests and error bounds indicate that these errors would likely be insignificant. However, as a further consideration, it was decided that a validation procedure would be useful as an additional check on the estimation procedure.

A method for validation was the primary motivation for the development of the exact procedures discussed in Section 4.1.1. After calculating exact levels of significance for the 2×2 table for a selected number of sample sizes, these can be compared, in some fashion, with the estimated levels from the Monte Carlo procedure.

With the number of tables generated for the calculation of each estimate as high as 2000, the normal distribution, as noted in the previous section, provides an excellent approximation to the binomial random variable X (number of rejections in $K=2000$ tables), where the binomial parameter p is the "exact" level of significance. Then $(X-Kp)/(Kp(1-p))^{1/2}$ has an approximate normal $(0,1)$ distribution. Dividing by K in the numerator and denominator, $(X/K-p)/(p(1-p)/K)^{1/2}$ has an approximate normal $(0,1)$ distribution. Forming a probability statement in a manner similar to the derivation of confidence intervals,

$$P(p - Z_{\alpha/2}(p(1-p)/K)^{1/2} < X/K < p + Z_{\alpha/2}(p(1-p)/K)^{1/2}) = 1 - \alpha. \quad (4-2)$$

These intervals, based on the "exact" levels of significance (p), can be used to compare the Monte Carlo results.

The exact program for the 2×2 table was run for the 15 probability vectors of the complete design (Appendix B) and for sample sizes $N = 4(4)36$ and $40(8)56$. This data is presented in Appendix F. The Monte Carlo program was run for the complete probability design and $N = 20(4)36$ and $40(8)96$. This provided an overlap area of comparable data for all 15 vectors and eight sample sizes, $N = 20(4)36$ and $40(8)56$, for each of the nine statistic/nominal level combinations. The overlap Monte Carlo data is presented in Appendix G. Intervals were calculated based on (4-2) using the "exact" levels of significance from the exact program for probabilities, $(1-\alpha)$, of .90 and .95. The Monte Carlo data, realizations of X/K , were then checked to see if they fell within these intervals. Summary percentages of those falling outside these intervals are presented in Table 9.

Before discussing the results of the validation, it should be noted that each row of Table 9 only corresponds to observations for 15 vectors. Even though total percentages are given for each (α, N) and for each N , these percentages are still based on those same 15 vectors. Therefore, care should be taken when drawing conclusions.

There appears to be no specific trend with respect to sample size or statistic. The estimates perform equally well for each statistic as reflected in the NTOTALS for each (statistic, α) combination.

Table 9. Monte Carlo Validation (I)

MONTE CARLO VALIDATION (I)
 PERCENTAGES OUTSIDE .95
 PROBABILITY INTERVALS

	<u>ALPHA = .10</u>				<u>ALPHA = .05</u>				<u>ALPHA = .01</u>				<u>NTOTAL</u>
<u>N</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>T</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>T</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>T</u>	
20	6	13	0	6	0	0	0	0	6	0	0	2	2
24	6	6	0	4	0	0	6	2	13	6	0	6	4
28	0	0	6	2	0	0	0	0	0	0	6	2	1
32	0	0	0	0	0	0	0	0	6	6	0	4	1
36	6	6	6	6	6	13	13	11	6	6	0	4	7
40	0	0	6	2	13	6	13	11	20	13	6	13	8
48	0	0	0	0	0	0	0	0	13	13	0	8	2
56	0	0	0	0	13	13	6	11	6	6	6	6	5
NTOTALS	2	3	2	2	4	4	5	4	9	6	2	6	4

MONTE CARLO VALIDATION (I)
 PERCENTAGES OUTSIDE .90
 PROBABILITY INTERVALS

	<u>ALPHA = .10</u>				<u>ALPHA = .05</u>				<u>ALPHA = .01</u>				<u>NTOTAL</u>
<u>N</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>T</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>T</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>T</u>	
20	13	13	6	11	6	0	6	4	6	0	0	2	5
24	13	6	0	6	0	0	13	4	20	13	6	13	8
28	0	0	6	2	6	6	6	6	13	26	6	15	8
32	0	13	20	11	6	6	6	6	6	6	20	11	9
36	26	40	6	24	20	13	26	20	13	13	26	17	20
40	0	6	13	6	20	20	13	17	26	26	6	20	14
48	6	0	6	4	6	13	0	6	13	13	13	13	8
56	0	13	13	8	20	13	6	13	13	20	13	15	12
NTOTALS	7	11	9	9	10	9	10	10	14	15	11	13	11

K - Kullback
 P - Pearson
 G - GSK

T - Total for each (α, N)
 NTOTAL - Total for each N

NTOTALS - { Total for each (statistic, α)
 Total for each α
 Grand Total

Table 10. Monte Carlo Validation (II)

MONTE CARLO VALIDATION (II)
 PERCENTAGES OUTSIDE .95
 PROBABILITY INTERVALS

	<u>ALPHA = .10</u>				<u>ALPHA = .05</u>				<u>ALPHA = .01</u>				<u>NTOTAL</u>
<u>N</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>T</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>T</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>T</u>	
20	6	6	0	4	0	0	0	0	0	0	13	4	2
24	0	0	0	0	20	6	0	8	13	6	0	6	5
28	6	0	6	4	0	0	0	0	6	6	0	4	2
32	0	0	13	4	6	13	6	8	6	0	13	6	6
36	0	0	0	0	6	0	0	2	6	0	13	6	2
40	20	6	6	11	6	13	6	8	20	6	6	11	10
48	6	6	6	6	0	0	6	2	0	0	0	0	2
56	0	0	0	0	6	0	0	2	6	6	0	4	2
NTOTALS	5	2	4	3	5	4	2	4	7	3	5	5	4

MONTE CARLO VALIDATION (II)
 PERCENTAGES OUTSIDE .90
 PROBABILITY INTERVALS

	<u>ALPHA = .10</u>				<u>ALPHA = .05</u>				<u>ALPHA = .01</u>				<u>NTOTAL</u>
<u>N</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>T</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>T</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>T</u>	
20	6	6	13	8	0	0	0	0	6	13	13	11	6
24	0	0	6	2	26	6	0	11	13	13	6	11	8
28	13	6	13	11	0	0	6	2	6	6	6	6	6
32	13	20	13	15	6	20	26	17	13	6	13	11	14
36	6	6	6	6	6	6	6	6	6	6	20	11	8
40	26	20	13	20	13	13	6	11	20	20	13	17	16
48	6	6	6	6	0	0	6	2	6	0	0	2	3
56	6	6	0	4	6	6	0	4	6	6	20	11	6
NTOTALS	10	9	9	9	7	6	6	6	10	9	11	10	8

K - Kullback
 P - Pearson
 G - GSK

T - Total for each (α, N)
 NTOTAL - Total for each N
 NTOTALS - { Total for each (statistic, α)
 Total for each α
 Grand Total

These NTOTALS are based on the 15 vectors and eight sample sizes, or 120 observations. The NTOTAL for each sample size indicates a highly irregular pattern, which should be expected with only 15 observations if the procedure is performing properly. There does appear to be a trend with respect to the nominal alpha levels. As the nominal alpha decreases the percentages increase. This might be expected since these intervals were formed based on the normal approximation to the binomial. This approximation is known to be less accurate for smaller values of p . The grand total percentages of four and 11 for the respective .95 and .90 probability intervals correspond very well with what is expected.

There was concern for some of the relatively high percentages, particularly for the sample size $N=36$ where the NTOTAL percentage was 20 for the .90 probability interval. It was decided to replicate the Monte Carlo simulation for these same overlapping sample sizes. Table 10 provides the summary results. The NTOTAL percentage for $N=36$ is eight for the .90 probability. It is obvious that for the first validation run, significantly more outliers affected the $N=36$ calculations. Considering the data in Table 10, as well as the data in Table 9, there are still no apparent trends with respect to sample size or statistic. The nominal alpha trend noted in Table 9 is not as clear in Table 10. There is an actual decrease in percentages from the nominal alpha .10 to the nominal alpha .05 for the .90 probability intervals. The grand total percentages of four and eight for the respective .95 and .90 probability intervals again are very good.

In summary, these validation results indicate that no trends exist with respect to the parameters of the simulation, and that the

simulation provides reasonable results. However, it is apparent that care should be taken when drawing conclusions from the data since severe irregularities will occur due to outliers.

CHAPTER V

DATA ANALYSIS

This chapter will present a detailed analysis of the data from the exact program and the Monte Carlo simulation. The data directly from these programs will be called the "basic" data. The exact 2×2 basic data are presented in Appendix F. The complete data for the 2×3 table provide examples of the Monte Carlo basic data and are presented in Appendix H. The presentation of this data will be discussed, followed by a comparison with other studies and a brief analysis with respect to general trends. From this basic data, minimum and maximum significance levels for minimum cell expectation intervals (MCEI) will be reported for each of the table, hypothesis, statistic, and nominal level combinations. These levels, presented in Appendix I, will be analyzed as to general trends with particular concern for possible results across tables. Returning to the basic data, graphical plots of this data will be used to determine certain "minimum N" (N_m) based on a confidence interval criterion. These N_m , presented in Appendix J, will provide the parameters for calculating Critical Expected Value (CEV) Distributions. These CEV distributions, presented in Appendix K, will be extensively analyzed. Comparisons will be made of the performance of the statistics, and trends indicated across tables. A useful parameter will be discussed along with other attempts of parametric analysis.

5.1 Basic Data

The presentation of the basic data (Appendices F and H) includes the underlying probability vector (P VECTOR), the sample size (N), and the exact or estimated exact levels of significance for the Kullback, Pearson, and GSK statistics at the nominal alpha levels of significance (.10, .05, .01). Each probability vector was used with all sample sizes of the design in Appendix C, and the vectors are arranged in the order of the scheme given in Appendix B. This section will first discuss some comparisons of the basic data with previous studies and then present a brief of analysis of some apparent trends.

5.1.1 Comparisons with Previous Studies

As discussed in Chapter II, previous studies were somewhat limited in scope, particularly with respect to the underlying probability structure. However, comparisons can be made with the study of this thesis for the basic data at certain points of the design. These comparisons are useful in further validating the procedure of this study and providing information for comparing conclusions.

The first study that provides some points for comparison is Haynam and Leone (1965). They presented graphs for the exact distribution of the Pearson X^2 statistic when used in testing independence for 2×3 and 3×3 tables for the multinomial sampling model, equiprobable vector, and sample sizes of 10 and 15. Using the Haynam and Leone graphs, Table 11 shows comparison points with the basic data from the Monte Carlo programs of this study. It should be noted that Haynam and Leone did not specify their procedures for handling zero marginals; although, for the equiprobable table the probability of zero marginals

Table 11. Comparisons with Haynam and Leone Data

2 × 3 Table

<u>Nominal α</u>	<u>Haynam and Leone</u>		<u>Kolb</u>	
.01	N=10	.004	N=12	.0025
	N=15	.006	N=18	.0085
.05	N=10	.048	N=12	.0490
	N=15	.050	N=18	.0465
.10	N=10	.100	N=12	.0975
	N=15	.100	N=18	.0970

3 × 3 Table

<u>Nominal α</u>	<u>Haynam and Leone</u>		<u>Kolb</u>	
.01	N=10	.004	N=18	.0060
	N=15	.006		
.05	N=10	.035	N=18	.0390
	N=15	.040		
.10	N=10	.075	N=18	.0950
	N=15	.095		

for the 2×3 and 3×3 tables would be relatively small. Table 11 demonstrates the close agreement between the exact levels of significance and the estimated levels from the Monte Carlo procedure of the present study.

Craddock (1966) provided estimated percentiles for the Pearson statistic under the independence hypothesis in 3×3 equiprobable tables, based on a Monte Carlo of 10,000 values with sample sizes ranging up to 100. Although exact comparisons are not possible due to the form of presentation, comparisons can be made as to whether the statistic is

conservative or liberal at the particular sample points. Table 12 provides these comparisons. Only at the .05 level are discrepancies

Table 12. Comparisons with Craddock Data

3 × 3 Table

<u>Craddock</u>				<u>Kolb</u>			
<u>Nominal α</u>				<u>Nominal α</u>			
<u>N</u>	<u>.10</u>	<u>.05</u>	<u>.01</u>	<u>N</u>	<u>.10</u>	<u>.05</u>	<u>.01</u>
15	C	C	C	18	C	C	C
20	-	C	C	27	L	L	C
25	L	C	C	36	L	L	C
30	L	C	C	108	L	C	L
100	L	L	L				

L - Liberal
C - Conservative
- - Equal

noted. Upon examination of the specific data at these points, the estimates are so close to the nominal levels that discrepancies can be interpreted to be the result of sampling error.

Craddock and Flood (1970) extended Craddock's study, including tables from 2×3 to 5×5 . They provided similar estimated percentiles for the Pearson X^2 . Table 13 reflects the comparisons with the present study. The percentiles given by Craddock and Flood were only estimated to one decimal place, so comparisons with the present study are difficult when exact levels are relatively close to nominal levels. Upon examination of those points considered exact (-) by Craddock and Flood, the largest error for the present study is seven percent of the nominal

Table 13. Comparisons with Craddock and Flood Data

Table	<u>Craddock and Flood</u>				<u>Kolb</u>			
	<u>Nominal α</u>				<u>Nominal α</u>			
<u>2 × 3</u>	<u>N</u>	<u>.10</u>	<u>.05</u>	<u>.01</u>	<u>N</u>	<u>.10</u>	<u>.05</u>	<u>.01</u>
	12	L	C	C	12	C	C	C
	18	L	L	C	18	C	C	C
	25	L	L	C	24	C	C	C
	30	L	L	C	30	L	C	C
	35	L	L	C	36	L	L	C
	50	L	L	C	48	L	L	L
<u>2 × 4</u>	16	C	C	C	16	C	C	C
	25	L	C	C	24	L	C	C
	30	L	-	C	32	L	-	L
	40	L	-	C	40	C	C	C
<u>2 × 5</u>	20	C	C	C	20	C	C	C
	30	-	C	C	30	L	C	C
	40	-	C	C	40	L	L	C
	50	-	C	C	50	L	L	C
<u>3 × 3</u>	18	-	C	C	18	C	C	C
	25	L	C	C	27	L	L	C
	35	L	C	C	36	L	L	C
	40	-	C	C	45	C	C	C
	50	-	-	C	54	L	C	C

level, and all estimates are well within a 95 percent confidence interval about the nominal level. There are then only 13 out of 57 points with apparent discrepancies, but upon closely examining these points the discrepancies are again apparently due to sampling error.

Miller (1979) estimated exact significance levels for both the Pearson and Kullback statistics for selected tables, probability vectors, and sample sizes based on a Monte Carlo study with 2000 tables. Those values comparable with the present study are given in Table 14. Comparisons for the 2×2 table for small N were excluded since exact values

Table 14. Comparisons with Miller Data

2 x 2 Table: N=80

<u>α</u>	<u>Miller</u> (.06,.19,.19,.56)		<u>Kolb</u> (.09,.21,.21,.49)	
	<u>Pearson</u>	<u>Kullback</u>	<u>Pearson</u>	<u>Kullback</u>
.10	.099	.103	.0995	.1005
.05	.047	.056	.0470	.0525
.01	.006	.012	.0105	.0115

2 x 5 Table: Equiprobable, N=20, 50

<u>α</u>	<u>Miller</u>		<u>Kolb</u>	
	<u>Pearson</u>	<u>Kullback</u>	<u>Pearson</u>	<u>Kullback</u>
.10	.091	>.180	.0890	.1845
.05	.030	.099	.0330	.1000
.01	.001	.019	.0020	.0185
.10	.091	.115	.1005	.1310
.05	.044	.058	.0465	.0710
.01	.004	.011	.0065	.0215

are available in the present study, but it should be noted that Miller's values compare very favorably. Table 14 indicates excellent agreement except for N=50 and the Kullback statistic. The discrepancy reflects the problem that can occur with outliers. Care must be taken not to draw conclusions based on only a few specific points. Trends can only be determined after the examination of a significant number of points.

These are the only possible comparisons with respect to the previous studies. They indicate a reasonable agreement, but also emphasize that these Monte Carlo results are subject to error and need to be con-

sidered carefully. Even with only these few comparisons it should be evident that there is a consistency with respect to the conservative or liberal nature of these estimates when the exact values are clearly on one side of the nominal levels. It should also be evident that when the estimate is near the nominal level, comparisons cannot be reasonably made.

5.1.2 Analysis

The basic data in Appendix H are representative of the data from the Monte Carlo simulation. Directly from the basic data, several interesting observations can be made. First, as should be expected, due to the asymptotic nature of these statistics, the general trend as the sample size N increases is for the estimated exact levels of significance to approach the nominal levels. However, the statistics each behave differently with respect to their asymptotic nature.

The Kullback statistic, at selected probability vectors, actually appears to depart from the general asymptotic trend. In fact, for some of the more extreme vectors (more skewed away from equiprobable) the estimated exact levels actually diverge from the nominal levels. Considering the 2×3 data in Appendix H, for the fourth probability vector, (.02, .04, .04, .18, .36, .36), at $N=12$ the estimated exact levels are .0915 and .0410 for the .10 and .05 nominal levels, respectively. These values are within 95 percent confidence intervals about the nominal levels. At $N=60$ the estimates are .1465 and .0730, well outside the confidence intervals. At $N=96$ the estimates are .1245 and .0665, still outside the confidence intervals. Figure 13 provides a graph of the estimates for the Kullback statistic under the probability vector. The

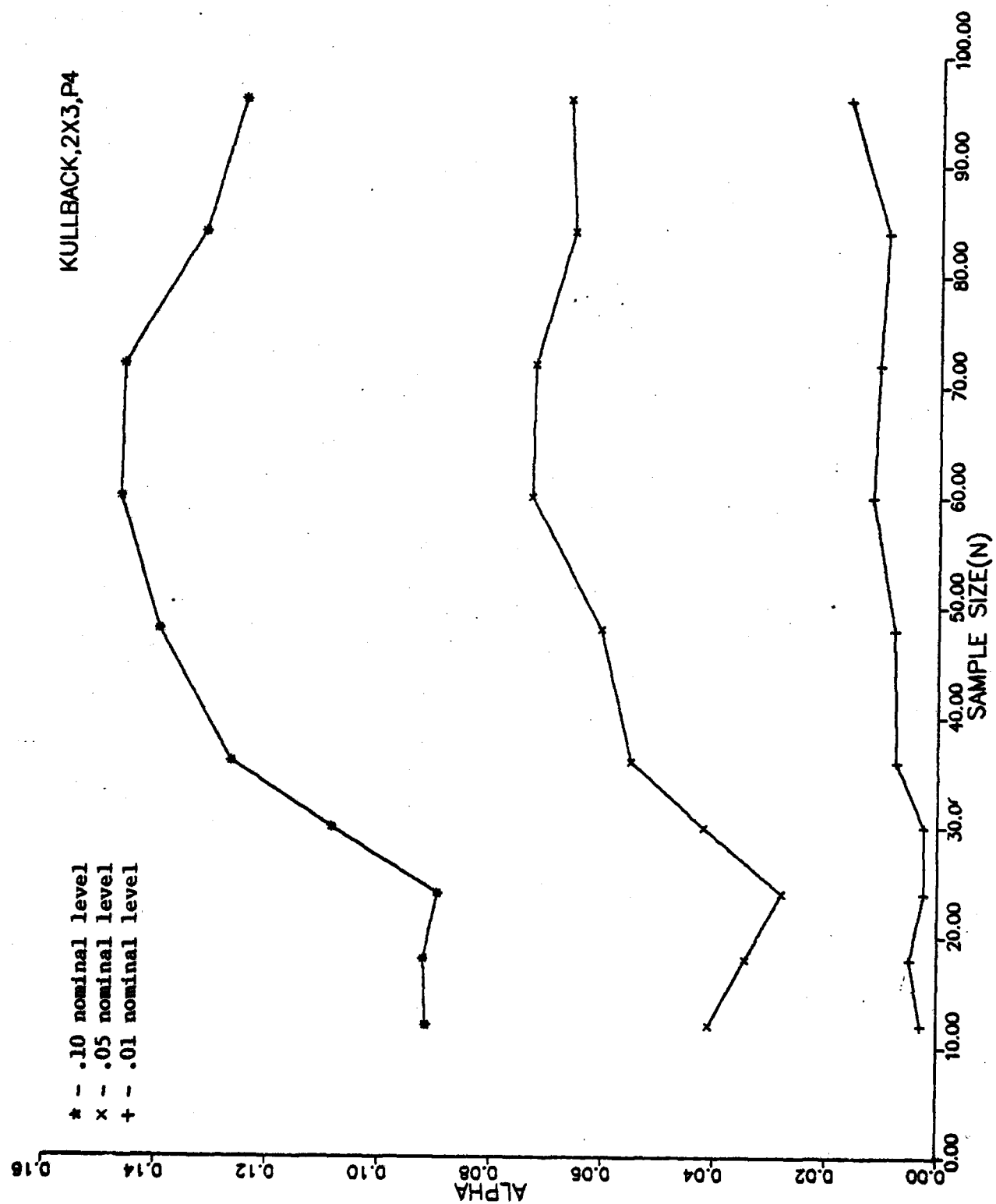


Figure 13. Kullback, 2 x 3, Probability Vector No. 4

graph demonstrates a quadratic effect for these estimates. For small N the estimates begin relatively close to the nominal levels, and then move farther away as N increases. At some N the estimates begin to move back toward the nominal levels. For the more extreme vectors this quadratic effect seems fairly consistent for all the tables and hypotheses considered. An additional general observation for the Kullback statistic is that it tends to be liberal. This observation agrees with the results from previous studies for the log-likelihood ratio statistic.

The GSK statistic is consistently conservative. In fact, for the more extreme vectors and larger tables the statistic is so conservative that its value as an independence hypothesis test statistic under the log-linear model is highly questionable. For these cases the power of the test would be so low that the null independence hypothesis would not be rejected even for extreme departures from independence. The GSK statistic does not begin to give reasonable results until near the equiprobable vector, and then only for large values of N .

One of the problems associated with the GSK statistic is the effect of zero observed cells as discussed in Section 4.1.2. The method for handling these zero cells was to add $1/r$ (r equal to the number of cells) to each zero cell. This method was chosen because it is the most popular. Little is known about the specific bias effect of this method or any of the other suggested methods. It is known that the bias tends to make the statistic more conservative. A case could be made that this is the reason for the ultra-conservative nature of the GSK statistic, demonstrated by the results of this study. This is probably valid

for the extreme vectors, where the probability of having sampling zeroes is relatively high, but this should not significantly affect the more equiprobable vectors. Table 15 provides some conditional probabilities for obtaining a zero observed cell, given the no zero marginal sampling procedure of this study. From this table it is obvious that a zero cell bias could significantly affect the estimates at the more extreme vectors, but should not significantly affect the estimates at the more equiprobable vectors.

Table 15. Conditional Zero Cell Probabilities

<u>Table</u>	<u>P VECTOR</u>	<u>N</u>	<u>Probability</u>
2 × 2	(.01,.09,.09,.81)	20	.8366
		56	.5645
		96	.3810
	(1/4,1/4,1/4,1/4)	20	.0127
		56	4.03×10^{-7}
		96	4.05×10^{-12}
2 × 3	(.01,.01,.08,.09,.09,.72)	24	.9608
		54	.8098
		96	.6185
	(1/6,1/6,1/6,1/6,1/6,1/6)	24	.0746
		54	.0003
		96	1.5×10^{-7}

The basic data for the Pearson statistic indicate that the statistic performs considerably different than either the Kullback or GSK statistic. With respect to the closeness of exact and nominal levels, the Pearson statistic is clearly superior to either of the other statistics, but it is also more erratic as to a conservative or liberal nature. For vectors close to the equiprobable the general trend of the statistic is to begin conservatively and move toward the

nominal level as N increases. For the more extreme vectors the general trend is to begin liberally and move toward the nominal levels. Exceptions to these trends do exist. In particular, for the most extreme vector, (.01, .01, .08, .09, .09, .72), of the 2×3 table at the nominal .05 level, the statistic shows a quadratic effect similar to the Kullback. The statistic begins conservatively and moves away from the nominal level, then turns abruptly (at $N=36$) back toward the nominal level. Also, at small values of N the statistic tends to be more erratic than the other statistics, but relatively quickly converges toward the nominal levels. The general superior performance of the statistic, as reflected in the basic data, corresponds to conclusions of other studies when comparing the Pearson and log-likelihood ratio statistics.

5.2 Minimum Cell Expectation (MCE)

One of the controversies surrounding the use of these asymptotic statistics has been how important minimum cell expectation (MCE) is with respect to the closeness of the chi-squared distribution to the exact distributions of these statistics. The regression example in Chapter I demonstrates the problems associated with small expected values, but in that example all the expected values become small.

5.2.1 Previous Results

Several authors have indicated that the MCE alone is significant; in particular, both Cramér (1946) and Cochran (1952) emphasized the importance of the MCE. Almost all the "rules of thumb" given for the use of the Pearson statistic are given in terms of MCE. However, very little empirical evidence is available that directly relates MCE and

the performance of any of these statistics for contingency table analysis.

Yarnold (1970) did present some empirical evidence for the Pearson statistic when used as a multinomial goodness-of-fit test. Here he provided his $5r/s$ rule. This study was somewhat contradicted by a study by Tate and Hyer (1973), where they concluded that, "general statements about minimum expectation are of limited usefulness". Only Odoroff's study provided some indication of the role of MCE with respect to contingency table tests. However, his MCE was not the usual contingency table expectation computed from the observed table. His MCE was based on choosing the parameters of his model so that the smallest cell expectation was constant. His results did indicate general improvement as MCE increased for most of the statistics he evaluated, but his results were restricted to the specialized framework of his model and hypothesis in the $2 \times 2 \times 2$ and $3 \times 2 \times 2$ tables with fixed and equal one-way marginals. None of the previous studies provided information regarding MCE and general independence test statistics.

As a means of comparing three competing multinomial goodness-of-fit statistics, Larntz (1978) used a very informative summary table. This table showed the minimum and maximum significance levels for nominal .05 tests using data from Monte Carlo sampling. These levels were reported based on the number of cells of the multinomial (2, 3, 5, 10) and intervals of minimum cell expectations. His results indicated general improvement of the statistics as MCE increased and provided a means for comparing the statistics.

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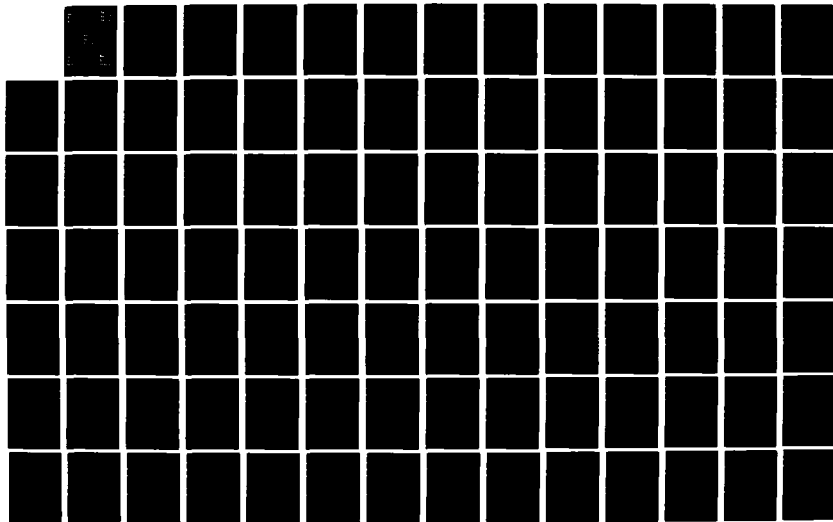
CURRENT METHODOLOGIES FOR THE ANALYSIS OF CONTINGENCY
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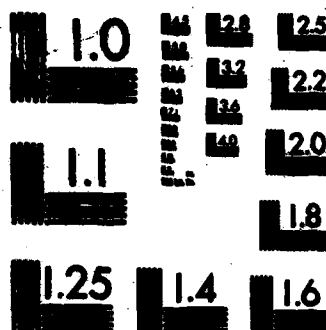
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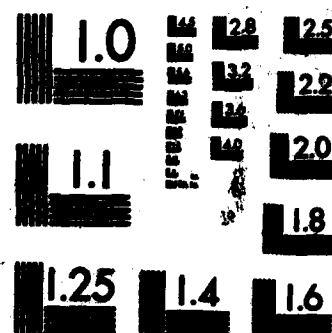
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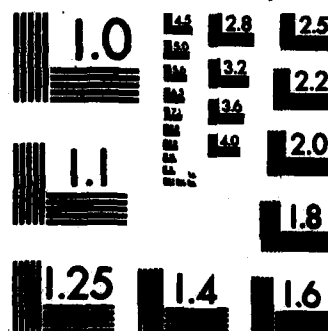




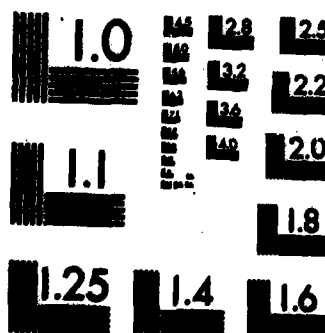
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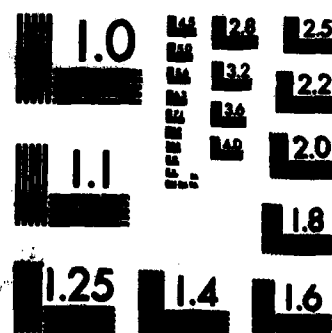
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5.2.2 Tabulation of Minimum and Maximum Significance Levels for Minimum Cell Expectation Intervals (MCEI)

Using the same procedure as Larntz, the basic data from the study of this thesis were used to tabulate minimum and maximum significance levels for selected minimum cell expectation intervals (MCEI). These MCEI were half-open/half-closed intervals, from $[0.5, 1.0)$ to $[4.5, 5.0)$, each of width 0.5. The levels were determined by considering each probability vector and sample size design point, then categorizing it into one of the MCEI by using the smallest cell probability and multiplying it by the sample size. The levels for each of the statistic and nominal level combinations were then compared against previous minimum and maximum levels for the corresponding MCEI. For each design point, minimum and maximum significance levels were updated as necessary until all design points were considered. These levels are presented in Appendix I for each of the table and hypothesis combinations and each of the statistic and nominal level combinations.

5.2.3 Analysis

With respect to these levels, performance of the statistics is measured by how close the minimum and maximum levels are to the nominal levels. Even a cursory look at these levels shows immediately that, in general, the statistics perform better as the MCE increases. Generalizations for the individual statistics follow very closely those of the previous section with respect to sample sizes.

The interval formed by the minimum and maximum significance levels (MMSLI) for the Kullback statistic generally includes the nominal level at the lowest MCEI, $[0.5, 1.0)$. As the MCEI increase, the MMSLI generally become smaller, but exclude the nominal level since the

minimum levels become larger than the nominal levels. This indicates the very liberal nature of the Kullback statistic.

The MMSLI for the GSK statistic are all relatively small, but the maximum level is always smaller than the nominal level. For lower MCEI the maximum levels are considerably smaller than the nominal levels. This reflects the ultra-conservative nature of the GSK statistic. There is an obvious improvement in the statistic as the MCE increases.

The MMSLI for the Pearson statistic are relatively large for the lower MCEI, but do include the nominal level. This reflects the erratic nature of the statistic for very small MCE. The MMSLI narrow as the MCEI increase and, in general, they continue to include the nominal level. For the higher MCEI both the minimum and maximum levels are generally both within a 95 percent confidence interval of the nominal levels. This is not true for either the Kullback or GSK statistics. Previous studies have indicated that the Pearson statistic could be used with expected values smaller than five. The results here further support this claim.

The more important aspect of this study is to investigate the nature of these statistics across tables. These minimum and maximum significance levels do provide valuable comparison information for the statistics, but trends across tables seem difficult to interpret. Only the levels for the GSK statistic indicate a trend. For a given MCEI, in general, these levels move farther away from the nominal levels (at .10 and .05) as the size of the table increases. This indicates a general degradation in the performance of the GSK statistic with

respect to MCE as the number of cells increases.

From the results here, MCE is obviously an important factor in the performance of these statistics. However, it does not appear to provide significant information to draw conclusions across tables, particularly for the Pearson and Kullback statistics. Also, with larger contingency tables, it would seem unlikely that just a consideration of one cell would be sufficient to measure the performance of a statistic. The next section will provide a means for finding trends across tables.

5.3 Critical Expected Value (CEV) Distributions

A significant aspect of any empirical study is a determination of appropriate parameters and "criteria" for measuring effectiveness. From the previous section it was seen that MCE alone did not provide sufficient information for measuring effectiveness in relation to the primary purpose of this study. This section will provide a criterion for measuring this effectiveness in the form of critical expected value (CEV) distributions.

5.3.1 Minimum Sample Size (N_m)

The primary concern regarding these statistics is their asymptotic nature; that is, given a specified underlying probability structure based on a null hypothesis of independence, when does the statistic begin to perform well. Here the answer is very subjective, and probably no single criterion would satisfy all situations. While this study is primarily concerned with discovering trends empirically under specified conditions, care needs to be taken to establish parameters and criteria that seem reasonable and that will provide trend

information.

Since these statistics are asymptotic, in that their exact distributions approach the chi-squared distribution as the sample size (N) increases, N would be a reasonable parameter. As far as a criterion for performance, some measure of closeness to the nominal level of significance would seem appropriate. The 95 percent confidence intervals; based on the Monte Carlo sampling of 2000 tables, the normal approximation to the binomial, and the nominal (.10, .05, .01) levels of significance; provide convenient and reasonable measures of "closeness" to these nominal values. These intervals, calculated exactly as the accuracy levels in Section 4.3.2, are (.0869, .1131), (.0404, .0596), and (.0056, .0144) for the nominal .10, .05, and .01 levels, respectively. The performance of a statistic will be considered good when the estimated exact levels are within these intervals, which are centered on the nominal levels. This criterion is intuitively appealing since a hypothesis test on the nominal level would not be rejected if the estimated exact level was within this interval.

The above criterion assumes that both overestimation and underestimation of the nominal levels are of equal importance. From the discussion in Section 4.1.1 this would seem reasonable since it provides equal consideration of both a conservative test and a more powerful test. Using intervals of this nature is not without precedence. Cochran (1952) suggested that a "disturbance" is unimportant if it lies within the intervals (.04, .06) and (.007, .015) for the nominal .05 and .01 levels, respectively. Roscoe and Byars (1971) used Cochran's intervals. Odoroff (1970) and Wang (1979) also used (.04,

.06) for the nominal .05 level. Wang chose an "acceptable" interval for the nominal .01 level of (.005, .015). Larntz (1978) used the interval (.03, .07) for the nominal .05 level. Which specific intervals are used is rather immaterial as long as they provide a means for comparing the performance of these statistics. It is unlikely that choosing any of the above intervals would lead to significantly different conclusions.

Once the intervals are selected, the goal is to determine for each statistic, table, hypothesis, nominal level, and probability vector combination the minimum sample size (N_m) where the exact significance levels are within the 95 percent confidence intervals and stay within these intervals as the sample size increases.

To assist in the determination of these N_m , graphs of the basic data were used. Figures 14, 15, and 16 are the graphs for the Kullback, Pearson, and GSK statistics, respectively, for the 2×3 table and fifteenth (P15) probability vector of the design in Appendix B. Each graph plots the estimated exact significance levels for each value of N . Each figure contains three graphs representing the three nominal levels using the following symbols:

- * - .10 nominal level
- × - .05 nominal level
- + - .01 nominal level

Straight lines are used to connect the points but have no special significance. They are used only as a visual guide to help identify out-

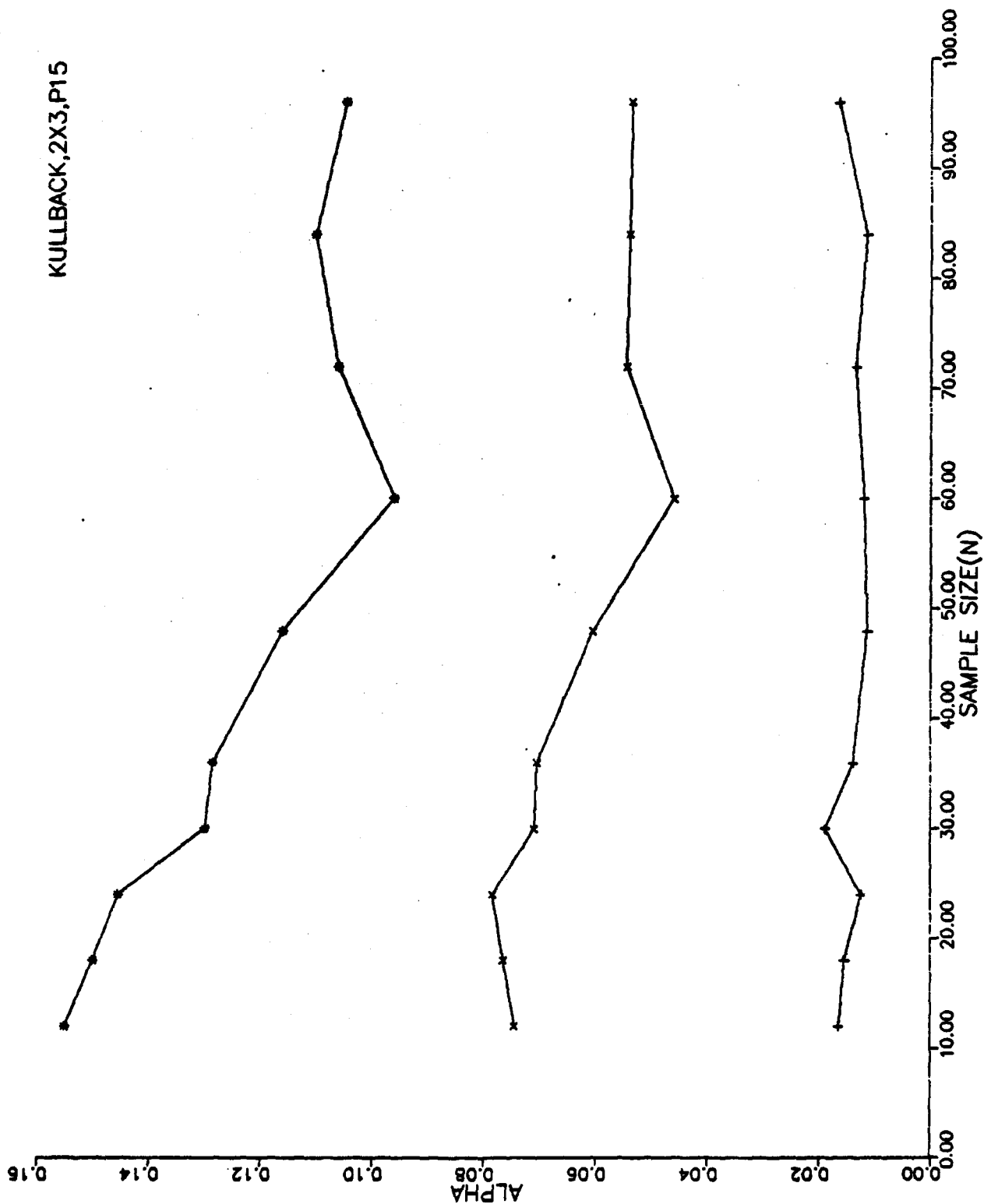


Figure 14. Kullback, 2 x 3, Probability Vector No. 15

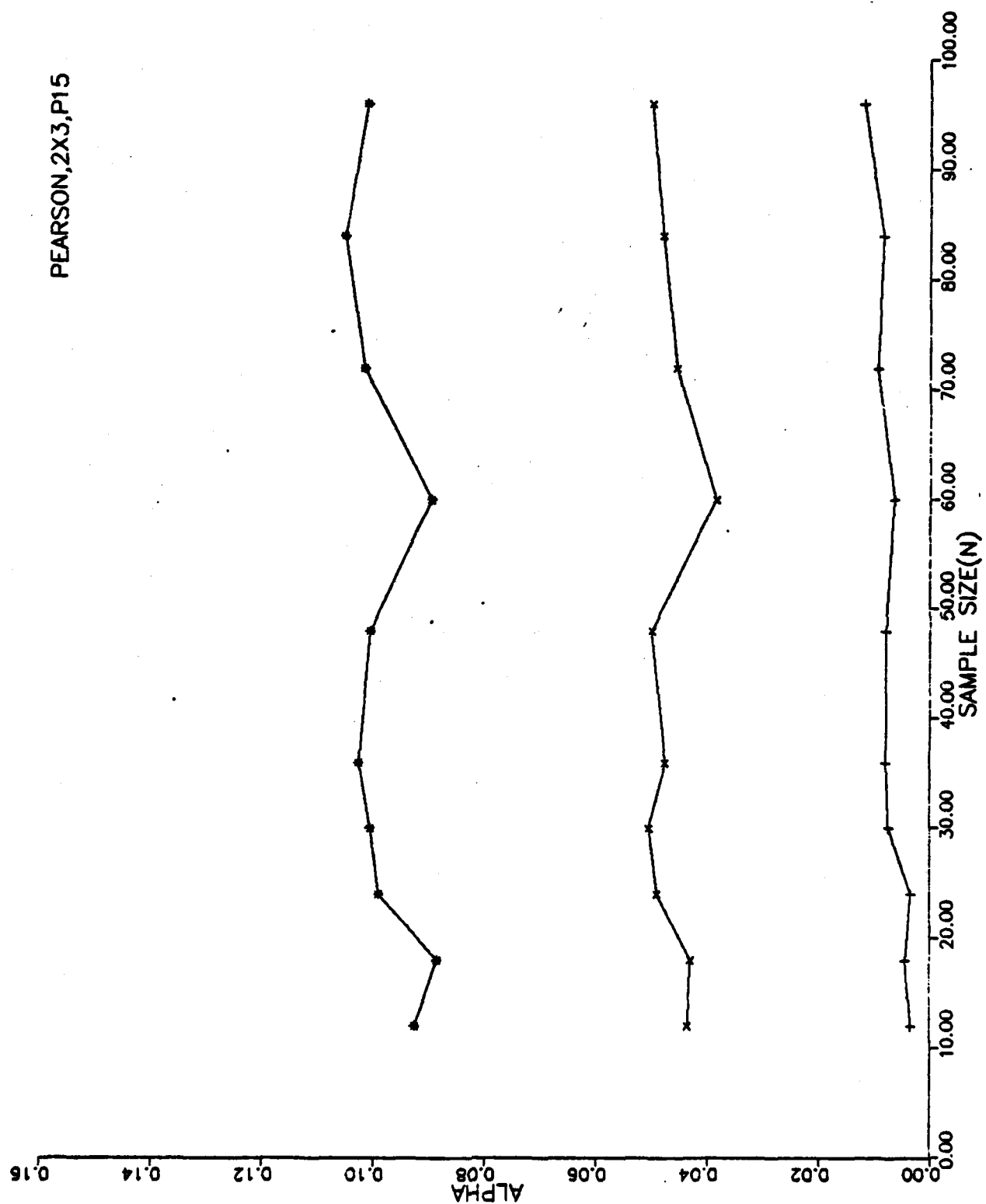


Figure 15. Pearson, 2 x 3, Probability Vector No. 15

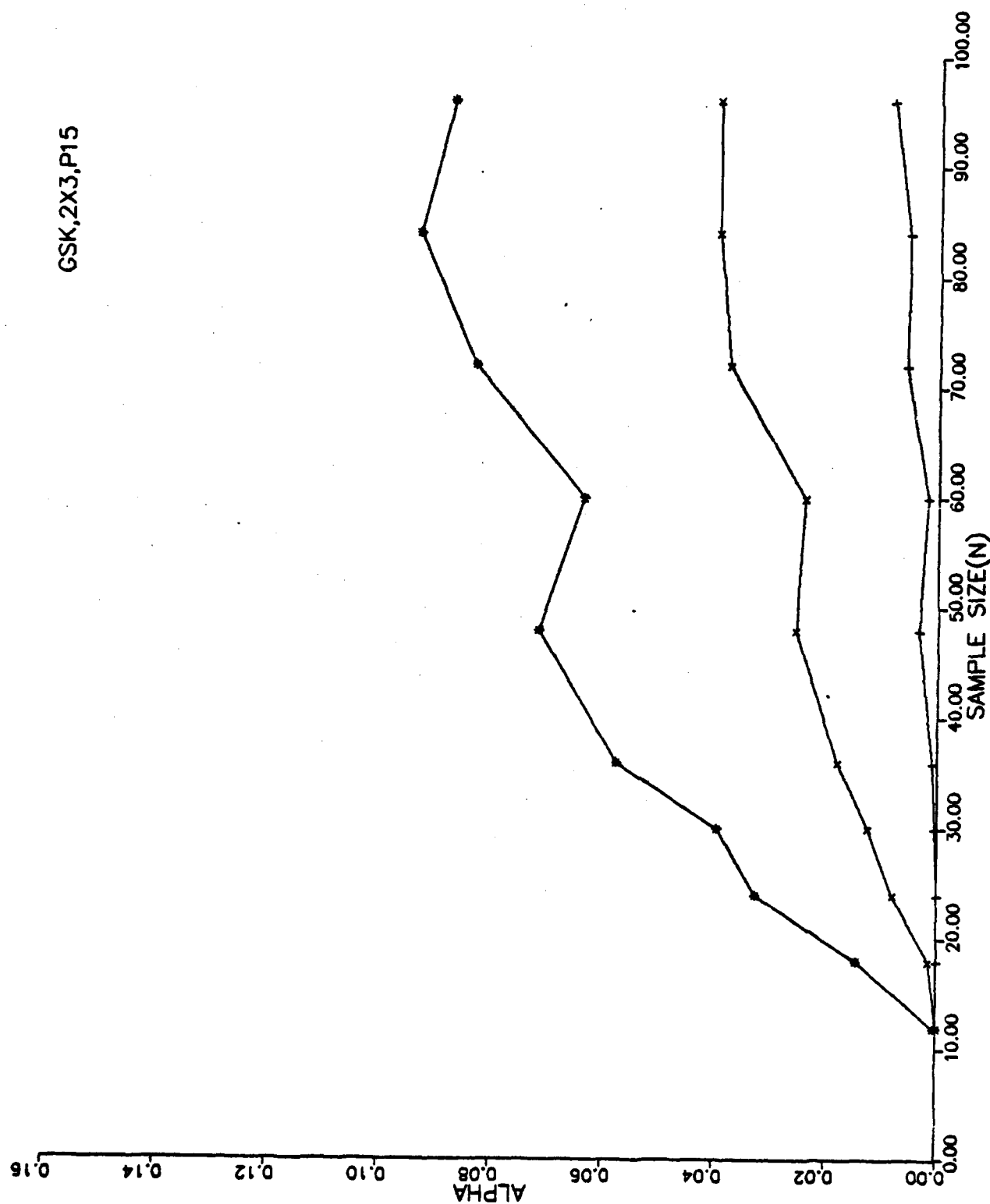


Figure 16. GSK, 2 x 3, Probability Vector No. 15

liers and determine the N_m .

The procedure used to determine these N_m was consistent and allowed for resampling of suspected outliers, particularly when these suspected outliers would cause choices of N_m which seemed drastically out of pattern with those around them. It should be noted that resampling did not always substantiate these points as outliers. In those cases the points were considered valid, and the resulting N_m reported. Fortunately, few of these situations occurred. The basic procedure is outlined below:

1. Fit the 95 percent confidence interval about the nominal level (Figure 17 - 95 percent confidence interval template).
2. Observe the pattern of data and note if there are any significant perturbations.
3. If the data points are all outside the confidence interval, report "greater than the maximum N " (e.g., >96 for the 2×2 table).
4. If the data points are all inside the confidence interval, report "less than the minimum N " (e.g., <12 for the 2×3 table).
5. If the data is relatively smooth, crossing into the confidence interval at fairly definitive points and staying within this interval, report this crossing sample size (approximately the linear interpolation between the last point outside the interval and the first point inside the interval).
6. If the data is not smooth and crossover points are not clear, indicate suspected outliers. Compare the data patterns with the patterns of other statistics under the same probability vector and, consequently, based on the same Monte Carlo sample. Designate suspected

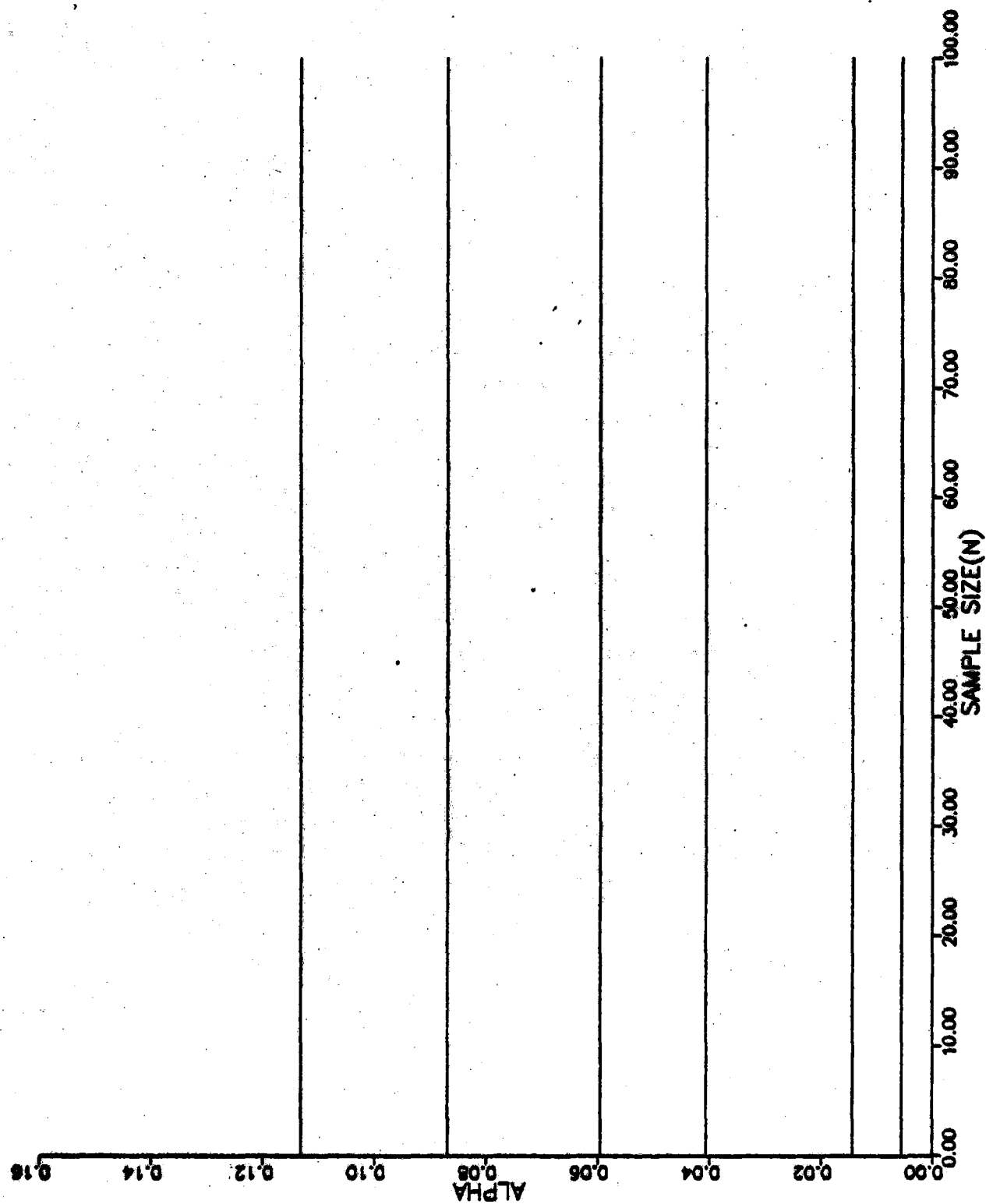


Figure 17. 95% Confidence Interval Template

outliers for resampling.

7. After steps one through six have been performed for each vector, statistic, and nominal level combination of a table, review to see if any N_m appear significantly out of pattern. If so, see if resampled points could clarify these determinations. Designate further points for resampling.

8. Resample all points designated in steps six and seven.

9. Average the resampled data with the basic data and return to step one until all N_m determinations have been satisfactorily made.

The outliers in step six were usually easy to determine. Figures 14, 15, and 16 provide a typical example. The point $N=60$ appears to be an outlier for all three statistics. Upon resampling, the estimated significance levels were considerably higher. Only one resampling run was needed for each table. For almost all cases suspected outliers were confirmed, and resampling smoothed the data curves, allowing for reasonable N_m determinations.

A problem did occur with the extreme vectors for the Kullback statistic. It appeared that for some extreme vectors the exact levels of significance were within the confidence intervals for relatively small N and would stay within these intervals. The actual situation was that these exact levels were drifting out of the confidence intervals, and the sample sizes were not large enough to detect this slow drift. This was verified for several of the extreme vectors by running the Monte Carlo programs for these vectors with larger sample sizes. Table 16 provides an example of this problem for the 2×4 table and an extreme vector at the .05 nominal level. From the original data sampled

Table 16. Kullback Extreme Vector Example

2 × 4 Table

Vector = (.01, .01, .03, .05, .09, .09, .27, .45)

Nominal α = .05 $\hat{\alpha}$ = estimated exact alpha

<u>N</u>	<u>$\hat{\alpha}_e$</u>	<u>N</u>	<u>$\hat{\alpha}_e$</u>
16	.0340	80	.0510
24	.0255	96	.0585
32	.0365	112	.0615
40	.0370	144	.0575
48	.0470	176	.0640
64	.0490	208	.0720

up to $N=96$, it would be easy to conclude that N_m occurred near $N=43$. However, with the extra data points it is clear that N_m is greater than 96 and even greater than 204. Other extreme vectors followed a similar pattern, and designating $N_m > 96$ for these extreme vectors corresponded to the general trend of larger N_m for more extreme vectors.

Appendix J provides tables of these N_m for each of the contingency tables and hypothesis tests considered. These N_m are listed in the probability vector order designated in Appendix B. For these N_m determinations the superiority of the Pearson statistic is obvious, and the Kullback statistic is clearly better than the GSK. However, the primary purpose of these N_m is not for comparison of the statistics but to provide parameters for the calculation of the critical expected value (CEV) distributions.

5.3.2 Calculation of CEV Distributions

Minimum N determinations alone are not useful to establish patterns across tables. A measure is needed to determine how well these

statistics perform with respect to small expected values, that can be compared across tables. Using the minimum N in Appendix J, a series of critical expected value (CEV) distributions were calculated and are presented in Appendix K. To calculate these CEV distributions, the N_m for a probability vector was multiplied by each of the cell probabilities. The corresponding expected values were then accumulated based on the number of expected values less than the integer values one through 10.

Each table of Appendix K presents the cumulative distribution of expected values at the minimum N for a given statistic, table, hypothesis, and nominal significance level combination using all the probability vectors of the design. The distributions in the upper part of each table are the "most extreme" expected value distributions for the probability vectors where the exact significance levels (as estimated by the Monte Carlo procedure) of the statistics are within the 95 percent confidence interval of the nominal level. "Most extreme" implies that the exact significance levels of the statistics for all less extreme distributions for these probability vectors are also within the 95 percent confidence interval. The lower part of each table corresponds to minimum N values that were determined to be greater than the maximum N used in the simulation. These distributions represent expected value distributions where exact significance levels of the statistic were not within the 95 percent confidence intervals.

5.3.3 Analysis

As will be discussed, summary parameters appear useless with respect to comparisons across tables. The CEV distributions do provide

a convenient method for comparing the performance of these statistics with respect to small expected values across tables. Before discussing these comparisons, it should be re-emphasized that these distributions were derived using the 95 percent confidence interval criterion. The upper part of the tables present the most extreme expected value distributions that meet this criterion where all other less extreme distributions also meet the criterion. These distributions then reflect the "best" expected value structures (smallest expected values and largest number of small expected values) where the statistic satisfies the given criterion. The lower part of the tables specifies expected value structures where the statistic does not satisfy the criterion. An advantage that these tables have over the MCE criteria is that they reflect not only the smallest expected value but also the entire expected value structure. In this respect they provide more information for comparison.

First, the performance of each statistic will be analyzed across tables in the order GSK, Kullback, and then Pearson, then the statistics will be compared. The tables are arranged in order of increasing cell size for each statistic and nominal level combination, with two-way tables preceding three-way tables and all tables grouped for each statistic.

The performance of the GSK statistic is the easiest to interpret. As previously discussed, the basic data reflected the highly conservative nature of this statistic. The estimated exact significance levels were considerably below the nominal levels except for equiprobable vectors at large sample sizes. Beginning with the two-

way tables and comparing the CEV distributions as the number of cells increases, it is quickly apparent that the performance of the GSK statistic with respect to small expected values significantly deteriorates. For the 2×2 table the upper CEV distributions include some expected values less than five for the .10 nominal level and some less than six for the .05 and .01 nominal levels. For the 2×3 table only values less than six are included for the .10 level and values less than seven for the .01 level. For the 2×4 table no values less than 10 were included at any nominal level. CEV distributions were not provided for the 2×5 table since no N_m were found less than the maximum sample sizes used. Monte Carlo estimations were not made for the GSK statistic in the 3×3 table due to its obvious deteriorating performance and the relatively extensive computational requirements.

Considering now the three-way tables, the Monte Carlo estimations were made for the GSK statistic in the $2 \times 2 \times 2$ table under both the no second-order interaction hypothesis and the complete interaction hypothesis. Estimations were not made for the $2 \times 2 \times 3$ table for the same reasons noted above for the 3×3 table. Comparing the results when moving from the one degree of freedom, no-second order interaction test to the four degree of freedom, complete interaction test, the performance of the GSK statistic is significantly degraded at all the nominal levels of significance. For the four degree of freedom test, only the equiprobable vector yielded N_m within the maximum sample sizes, and these N_m were estimated to be equal to these maximum values.

The results above indicate that for a given independence test the GSK statistic is less robust with respect to small expected values as

the number of cells increases, and that within a given size table the statistic is less robust as the degrees of freedom of the test increases. Also, the statistic is ultra-conservative and ineffective for most independence hypothesis testing situations under the log-linear model.

Results for the Kullback statistic are much more difficult to interpret and do present some inconsistencies. First, considering the two-way tables at the .10 nominal level, from the 2×2 to the 2×3 table there is very little change in the performance of the statistic. The minimum expected value (MCE) in the upper part of the CEV distributions for the 2×2 table is less than four, and there are four separate vectors that have that small a value. For the 2×3 table there is one vector with a MCE less than three and one additional vector with a MCE less than four. However, there are two vectors in the lower part of the 2×3 table (where the statistic does not satisfy the criterion) that have CEV distributions with MCE just less than four. Here there is an inconsistency, where some vectors indicate improvement in performance and others indicate a degradation. The 2×3 table does contain a "richer" CEV structure than the 2×2 table at the higher ends of the upper distributions, but this may be offset by more vectors in the lower distributions for the 2×3 than the 2×2 table. Moving to the 2×4 table, the distributions indicate a slightly poorer performance for the statistic. For the 3×3 and 2×5 tables the distributions indicate very little change. Similar results are obtained at the .05 level.

Results at the .01 level are very inconsistent. This may be caused in part by the difficulty in making some N_m determinations for

the statistic at the .01 level. As previously mentioned the quadratic nature of the Kullback statistic affected the N_m determinations at the extreme vectors. This was especially critical at the .01 nominal level. For the most extreme vectors, determinations of N_m greater than maximum sample sizes were relatively easy to make. For the mean equiprobable vectors, quadratic effects were not a problem. However, for the middle vectors it was difficult to determine if the exact significance level would continue within the confidence interval as N increased beyond the maximum N . Therefore, the CEV distributions for these vectors appear less stable. At this .01 level the inconsistencies presented seem to indicate neither general improvement nor degradation. Comparing the 2×2 and 2×5 tables, the upper CEV distributions are "richer" for the 2×5 table for the larger expected values, but for expected values less than five, four, etc., there is very little difference. The lower CEV distributions provide no additional information.

These results indicate that the robustness of the Kullback statistic with respect to small expected values does not change as the number of cells increase.

Now considering the three-way tables, the performance of the Kullback statistic slightly improves from the one degree of freedom, no second-order interaction test to the four degree of freedom, complete independence test in the $2 \times 2 \times 2$ table. This improvement is more definite for smaller nominal levels. Moving to the seven degree of freedom, complete interaction test for the 12 cell, $2 \times 2 \times 3$ table, again there are inconsistencies which indicate no apparent trend at any of the nominal levels. These results agree with those of two-way

tables with respect to increased number of cells for a given hypothesis test. In comparing the different hypothesis tests within the same size table, the Kullback statistic performs better for the higher degree of freedom test.

The results for the Pearson statistic are relatively easy to interpret and are very consistent. Initially considering the two-way tables, as the number of cells increases, the CEV distributions improve since there are smaller expected values and more small expected values where the statistic satisfies the given criterion. This is true at all the nominal levels of significance. More specifically, considering the .10 nominal level CEV distributions for the 2×2 table, the largest number of expected values less than one is two, and the largest number less than five is four. For the 2×3 table these numbers are three and six, respectively; for the 2×4 table, four and eight; for the 3×3 table, five and nine; and for the 2×5 table, five and ten. This improvement indicates that the Pearson statistic is more robust with respect to small expected values as the number of cells increases.

Now considering the CEV distributions for the three-way tables, the performance of the Pearson statistic improves at all nominal levels when moving from the one degree of freedom, no second-order interaction test to the four degree of freedom, complete independence test in the $2 \times 2 \times 2$ table. The performance further improves when moving to the seven degree of freedom, complete independence test in the $2 \times 2 \times 3$ table. These improvements are easily distinguished by noting the significant improvement in the "richness" of CEV structure. These results further demonstrate the improved robustness of the Pearson

statistic with respect to small expected values as the number of cells increase, and they indicate an improved robustness within a given size table for higher degree of freedom tests.

It should be noted that comparisons were not made across table dimensions. It does not seem reasonable to compare different "type" tests except within the same table size under the hierarchical structure of the log-linear model. However, comparisons could be considered between the two-way independence tests (complete independence for the two-way table) and the three-way complete independence tests. To be consistent these comparisons should be made with respect to the number of cells.

First, considering the eight cell, 2×4 and $2 \times 2 \times 2$ tables and the complete independence tests, the CEV distributions for all three statistics at all three nominal levels are remarkably similar, without exception. This would indicate that for a given type test the number of cells alone (regardless of table dimension) determines the performance characteristics for each statistic. It was previously shown for each statistic that results for the two-way and three-way tables were consistent with respect to the increase in number of cells. Considering now the 2×5 and $2 \times 2 \times 3$ tables with 10 and 12 cells, respectively, the results are again consistent for all three statistics at all three nominal levels. These observations indicate that within a "type" hypothesis the previous results can be extended to general statements with respect to the number of cells across table dimensions.

These CEV distributions provide an excellent means for comparing the performance of these statistics with respect to small expected

values. Clearly, the Pearson statistic is superior for all the two-way tables. Its exact levels of significance enter the 95 percent confidence intervals at smaller N (N_m) than the levels of either the Kullback or GSK statistic for every two-way table, vector, and nominal level combination. Therefore, the CEV distributions for the Pearson statistic are "richer", indicating the statistic is more robust with respect to small expected values. For the two-way tables the performance of the GSK statistic is only comparable to that of the Kullback statistic in the 2×2 table at or near the equiprobable vector. In fact, it is slightly superior at the .10 and .05 nominal levels. But, the performance of the GSK statistic quickly deteriorates away from the equiprobable vector and for larger tables. In these cases the Kullback statistic is clearly superior.

For the three-way tables the GSK statistic shows some strength for the one degree of freedom test of no second-order interaction in the $2 \times 2 \times 2$ table. Its performance is slightly better than those of either the Kullback or Pearson at the .10 level and comparable at the .05 and .01 levels. For the four degree of freedom, complete interaction test in the $2 \times 2 \times 2$ table, the GSK statistic performs very poorly, the Kullback statistic shows some improvement and is superior to the GSK, and the Pearson again dominates the other statistics. In the $2 \times 2 \times 3$ table the Pearson statistic is clearly superior since its performance improves from the $2 \times 2 \times 2$ table, while the performance of the Kullback statistic shows little change, and the performance of the GSK statistic would be expected to deteriorate.

5.2.3 μ_2 and Other Parameters

The individual vectors for the CEV distributions have been arranged in order of decreasing second moment, μ_2 , of the probability vectors. μ_2 is a measure of the spread of the probability vector and is calculated as

$$\mu_2 = \sum_{i=1}^k p_i^2/k - (1/k)^2, \quad (5-1)$$

where the p_i are the cell probabilities and k is the number of cells.

Interpreting from the calculations of N_m , the data in Appendix K indicate that the statistics generally perform better as the μ_2 parameter decreases. The best performance (smallest N_m) is near the equiprobable vector, where $\mu_2 = 0$. Several authors have previously suggested that equiprobable structures are best for the use of the Pearson statistic under small expected values [e.g., see Roscoe and Byars (1971)]. From this data it is interesting to observe the significant relationship between N_m and μ_2 for the Kullback and GSK statistics and to some extent for the Pearson statistic.

The μ_2 parameter not only provides an indication of a single small probability cell, but also provides a measure of the relative smallness of the probabilities of all cells. The independence structure requires that a number of small probability cells be balanced with a relatively equal number of large probability cells. Although there are perturbations, the data in Appendix K generally indicate that for a given table these statistics perform worse both as the smallest expect-

ed value decreases and as the number of small expected values increases.

Although μ_2 provides a convenient parameter for each individual table, comparisons across tables is difficult. Under multinomial sampling as the number of cells increases, the probability structure becomes "diluted". For the two-way tables under the independence hypothesis the maximum μ_2 approaches $1/k - 1/k^2$. Table 17 lists these maximum μ_2 for the two-way tables of this study. For a given sample size a more

Table 17. Maximum μ_2 : Two-Way Tables

<u>Table</u>	<u>k</u>	<u>max μ_2</u>
2 × 2	4	.1875
2 × 3	6	.1389
2 × 4	8	.1094
3 × 3	9	.0988
2 × 5	10	.0900

diluted structure creates more small expected values, thus minimum N calculations are not directly comparable.

It was thought that a correction for this dilution effect might be useful. The μ_2 values were "standardized" for each table by dividing by the maximum μ_2 values of Table 17,

$$\mu'_2 = \mu_2 / \max \mu_2. \quad (5-2)$$

An attempt was then made to compare the minimum N determinations for these standardized μ_2 across tables. These comparisons were highly

erratic.

Another parameter was considered that would account for the increase in table size. A coefficient of variation (V) was calculated using the ratio of the standard deviation of the expected values and the mean of the expected values. The variance of the expected values is $N^2\mu_2$, so that,

$$V = N(\mu_2)^{1/2}/(N/k) = k(\mu_2)^{1/2}. \quad (5-3)$$

Calculations were made and comparisons attempted across tables. Again the comparisons were very erratic.

Other parameters were considered, including the third and fourth moments of the probability structures and coefficients of skewness and kurtosis. Linear models with combinations of these parameters were also considered. None of these parameters or models provided consistent results. μ_2 was the best measure for performance within tables, but the effects across tables did not appear sufficiently stationary to model effectively. The CEV distributions provided the only reasonable vehicle to compare results across tables.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

As stated in the Introduction, the principal focus of this research was the investigation of current methodologies for the analysis of contingency tables with respect to the classic problem of small expected values. The more specific purpose of this research was to investigate the robustness characteristics of the associated statistics with respect to small expected values as the size of the table increases. A secondary purpose was to compare these statistics and make recommendations for their use. With these considerations this chapter will present the conclusions of this investigation, provide recommendations for the use of the statistics investigated, and make recommendations for future research.

6.1 Conclusions

From the results of this study the following conclusions can be drawn:

- 1) The basic data (Appendices F and H) reflect the asymptotic nature of these statistics. Generally, as the sample size increases, the exact levels of the statistics approach the nominal levels, indicating that the exact distributions approach the chi-squared distribution.
- 2) The minimum and maximum significance levels for minimum cell expectation intervals (MCEI) (Appendix I) demonstrate that the performance of these statistics within a certain size table generally improves

as the MCE increases. These levels do not provide reasonable comparisons across tables.

3) The second moment of the underlying probability vector, μ_2 , calculated from (5-1), is a convenient parameter for measuring the performance of a statistic within a certain size table. The μ_2 parameter is a measure of how "extreme" the underlying probability structure is with respect to the equiprobable vector, where $\mu_2 = 0$. The μ_2 parameter not only accounts for a single small cell probability but also a number of small cell probabilities. In general, as μ_2 decreases, the performance of these statistics improves. Each statistic performs best at or near the equiprobable vector.

4) The determination of minimum sample sizes (N_m) provides the parameters for calculating critical expected value (CEV) distributions. These CEV distributions provide a vehicle, not only for comparing the performances of these statistics, but also for determining trends of the robustness of these statistics with respect to small expected values as the size of the table increases.

5) The GSK statistic is highly conservative to the point of being ineffective as a test statistic for most of the hypothesis situations in this study, particularly at the more extreme probability vectors. The statistic appears to be significantly biased by the zero cell correction procedure, particularly at the more extreme probability vectors where the probability of obtaining a sampling zero is relatively high. The performance of the GSK statistic significantly deteriorates as the number of cells of the table increases even for the more equiprobable structures. The CEV distributions indicate that

the statistic is less robust with respect to small expected values as the size of the table increases.

6) The Kullback statistic generally tends to be liberal and demonstrates a "quadratic" asymptotic nature, especially at the more extreme probability vectors. The statistic does not perform nearly as well as the Pearson but does generally perform better than the GSK. The Kullback statistic performs best at the lowest nominal level, $\alpha = .01$. With respect to an increase in the number of cells, the performance of the statistic is inconsistent. In general, the Kullback statistic demonstrates little change in robustness with respect to small expected values as the size of the table increases.

7) Of the three statistics evaluated, the Pearson statistic is clearly superior in performance. In many cases it tends to be conservative, but its exact levels of significance approach the nominal levels relatively quickly as the sample size increases. The performance of the statistic significantly improves as the number of cells increases. The Pearson statistic is more robust with respect to small expected values as the size of the table increases.

8) The statistics were evaluated for the $2 \times 2 \times 2$ table for both the one degree of freedom, no second-order interaction hypothesis and the four degree of freedom, complete interaction hypothesis. From the one degree of freedom to the four degree of freedom test, the performance of the Pearson statistic significantly improves, that of the Kullback statistic slightly improves, and that of the GSK statistic deteriorates. These results indicate that, at least for hierarchical hypotheses, the Pearson and Kullback statistics are more robust and

the GSK statistic is less robust with respect to small expected values as the number of degrees of freedom of the test increases.

From these conclusions the following recommendations are made:

- 1) Unless a highly conservative test statistic is desired, the GSK statistic should not be used in hypothesis testing under the log-linear model structure. This recommendation is most important when the observed structure contains sampling zero cells, where some technique would be needed to correct for these cells.
- 2) The Pearson statistic should be preferred over the Kullback in these independence hypothesis testing situations. This is especially true for larger contingency tables.
- 3) The analyst should feel confident in the use of the Pearson statistic for large tables even when some expected values are very small.

6.2 Recommendations for Future Research

This study provided the first broad investigation of the robustness of these statistics with respect to small expected values as the size of the table increases. However, this study was limited to certain specific conditions including the tables, hypotheses, sampling model, and log-linear model. Recommendations for future research include:

- 1) Investigations of the robustness of these statistics with respect to other hypothesis tests and sampling models; in particular, the homogeneity hypothesis under product-multinomial sampling.
- 2) Investigation of the robustness of statistics using other underlying models and formations of interaction hypotheses; in particular, the performance of the GSK statistic using the linear model and

corresponding independence hypotheses.

3) A further investigation of the relationship of μ_2 , or associated parameters, and the performance of these statistics.

4) Investigation of the effect that "closeness" of the exact levels of significance to the nominal levels has on the power of the test under selected alternative hypotheses.

5) Investigation of the robustness of the GSK statistic using other methods for correction of sampling zero cells; in particular, the .05 correction of Goodman (1970) or the Bhapkar (1979) correction.

APPENDIX A

APPROXIMATIONS TO MDI STATISTICS

This appendix follows the derivations given in Gokhale and Kullback (1978, Appendix) except for the Maclaurin series expansion of (A-4).

Let $T_i(\omega)$ ($i=1,2,\dots,n$) be a set of linearly independent statistics defined over the set of cells Ω . The MDI theorem proved by Kullback (1959) gives the value of $p(\omega)$ which minimizes the discrimination information,

$$I(\underline{p}:\underline{\pi}) = \sum_{\Omega} p(\omega) \ln(p(\omega)/\pi(\omega)), \quad (\text{A-1})$$

between two distributions \underline{p} and $\underline{\pi}$ over the family, P of \underline{p} -distributions which satisfy

$$\sum_{\Omega} T_i(\omega)p(\omega) = \theta_i^*; i=1,2,\dots,n, \quad (\text{A-2})$$

for given θ_i^* . If $\pi(\omega)$ satisfies (A-2), then $I(\underline{p}:\underline{\pi}) = 0$. Otherwise, the MDI theorem gives

$$p^*(\omega) = \exp\left(\sum_{i=1}^n \tau_i T_i(\omega)\right) \pi(\omega) / M(\underline{\tau}), \quad (\text{A-3})$$

where

$$M(\underline{\tau}) = \sum_{\Omega} \exp\left(\sum_{i=1}^n \tau_i T_i(\omega)\right) \pi(\omega). \quad (\text{A-4})$$

The τ parameters are equivalent to Lagrangian multipliers whose values are defined in terms of the known θ_i^* ,

$$\begin{aligned} \theta_i^* &= \frac{\partial}{\partial \tau_i} M(\underline{\tau}) \\ &= \sum_{\Omega} [\exp(\sum_{i=1}^n \tau_i T_i(\omega)) T_i(\omega) \pi(\omega)] / M(\underline{\tau}); \\ &\quad i=1, 2, \dots, n. \end{aligned} \quad (\text{A-5})$$

Note that when $\tau_1 = \tau_2 = \dots = \tau_n = 0$, (A-3) implies that

$$p^*(\omega) = e^{0 \cdot \pi(\omega)} / \sum_{\Omega} e^{0 \cdot \pi(\omega)} = \pi(\omega). \quad (\text{A-6})$$

From (A-3) the familiar log-linear model of Chapter III can be written,

$$\ln[p^*(\omega)/\pi(\omega)] = L + \sum_{i=1}^n \tau_i T_i(\omega), \quad (\text{A-7})$$

where $L = -\ln M(\underline{\tau})$. The minimum value of the discrimination information in (A-1) is then

$$\begin{aligned}
I(\underline{p}^* : \underline{\pi}) &= \int_{\Omega} p^*(\omega) \ln(p^*(\omega)/\pi(\omega)) \\
&= \int_{\Omega} \{ [\exp(\sum_{i=1}^n \tau_i T_i(\omega)) \pi(\omega) / M(\underline{\tau})] [L + \sum_{i=1}^n \tau_i T_i(\omega)] \} \\
&= \sum_{i=1}^n \tau_i \{ \int_{\Omega} [\exp(\sum_{i=1}^n \tau_i T_i(\omega)) T_i(\omega) \pi(\omega) / M(\underline{\tau})] \\
&\quad + (L/M(\underline{\tau})) \int_{\Omega} \exp(\sum_{i=1}^n \tau_i T_i(\omega)) \pi(\omega),
\end{aligned}$$

and substituting from (A-4) and (A-5),

$$\begin{aligned}
I(\underline{p}^* : \underline{\pi}) &= \sum_{i=1}^n \tau_i \theta_i^* + (L/M(\underline{\tau})) M(\underline{\tau}) \\
&= \sum_{i=1}^n \tau_i \theta_i^* + L.
\end{aligned} \tag{A-8}$$

From Kullback (1959, 1970) the following duality relations exist between the θ and τ parameters and the covariance matrix of the $T_i(\omega)$. Letting

$$(\underline{d\theta}^*)' = (d\theta_1^*, d\theta_2^*, \dots, d\theta_n^*)$$

and

$$(\underline{d\tau})' = (d\tau_1, d\tau_2, \dots, d\tau_n),$$

then

$$\underline{d\theta}^* = \underline{\Sigma}^* \underline{d\tau}, \quad \underline{d\tau} = \underline{\Sigma}^{*-1} \underline{d\theta}, \quad (\text{A-9})$$

where $\underline{\Sigma}^*$ is the covariance matrix of the $T_i(\omega)$ with respect to the $p^*(\omega)$ distribution, and

$$\underline{\Sigma}^* = (\sigma_{ij}^*), \quad \sigma_{ij}^* = \int_{\Omega} (T_i(\omega) - \theta_i^*)(T_j^*(\omega) - \theta_j^*) \pi(\omega)$$

and

$$\underline{\Sigma}^{*-1} = (\sigma^{*ij}), \quad \frac{\partial \theta_i^*}{\partial \tau_j} = \sigma_{ij}^*, \quad \frac{\partial \tau_i}{\partial \theta_j^*} = \sigma^{*ij}. \quad (\text{A-10})$$

From (A-4) a Maclaurin series expansion can be used to derive an approximation for $\ln M(\underline{\tau})$ up to quadratic terms. Using $e^x \approx 1 + x + x^2/2!$ and substituting into (A-4),

$$\begin{aligned} M(\underline{\tau}) &\approx \int_{\Omega} [1 + \sum_i \tau_i T_i(\omega) + \frac{1}{2} (\sum_i \tau_i T_i(\omega))^2] \pi(\omega) \\ &= \int_{\Omega} \pi(\omega) + \sum_i \tau_i \int_{\Omega} T_i(\omega) \pi(\omega) \\ &\quad + \frac{1}{2} \sum_i \pi(\omega) (\sum_i \tau_i T_i(\omega))^2. \end{aligned}$$

Letting

$$\theta_i = \int_{\Omega} T_i(\omega) \pi(\omega), \quad (\text{A-11})$$

noting that $\sum_{\Omega} \pi(\omega) = 1$, and expanding the third term,

$$M(\underline{\tau}) \approx 1 + \sum_i \tau_i \theta_i + \frac{1}{2} \sum_{\Omega} \pi(\omega) \sum_i \sum_j \tau_i \tau_j T_i(\omega) T_j(\omega).$$

Now, using $\ln(1+x) \approx x - \frac{1}{2} x^2$ and disregarding terms larger than quadratic,

$$\ln M(\underline{\tau}) \approx \sum_i \tau_i \theta_i + \frac{1}{2} \sum_{\Omega} \pi(\omega) \sum_i \sum_j \tau_i \tau_j T_i(\omega) T_j(\omega) - \frac{1}{2} \left(\sum_i \tau_i \theta_i \right)^2.$$

Looking at the third term and expanding,

$$\left(\sum_i \tau_i \theta_i \right)^2 = \sum_i \sum_j \tau_i \tau_j \theta_i \theta_j.$$

Adding and subtracting $\theta_i \theta_j$,

$$\left(\sum_i \tau_i \theta_i \right)^2 = \sum_i \sum_j \tau_i \tau_j (\theta_i \theta_j + \theta_i \theta_j - \theta_i \theta_j).$$

Using Equation (A-11), substituting

$$\theta_j = \sum_{\Omega} T_j(\omega) \pi(\omega)$$

in term one and

$$\theta_i = \sum_{\Omega} T_i(\omega) \pi(\omega)$$

in term two, and multiplying by $\sum_{\Omega} \pi(\omega)$ in term three,

$$\left(\sum_i \tau_i \theta_i\right)^2 = \sum_{\Omega} \pi(\omega) \sum_i \sum_j \tau_i \tau_j (\theta_i T_j(\omega) + \theta_j T_i(\omega) - \theta_i \theta_j).$$

Substituting into the above $\ln M(\underline{\tau})$ approximation,

$$\begin{aligned} \ln M(\underline{\tau}) \approx & \sum_i \tau_i \theta_i + \frac{1}{2} \sum_{\Omega} \pi(\omega) \sum_i \sum_j \tau_i \tau_j (T_i(\omega) T_j(\omega) - \theta_i T_j(\omega) \\ & - \theta_j T_i(\omega) + \theta_i \theta_j). \end{aligned}$$

Letting

$$\sigma_{ij} = \sum_{\Omega} (T_i(\omega) - \theta_i)(T_j(\omega) - \theta_j) \pi(\omega); \quad \begin{matrix} i=1,2,\dots,n; \\ j=1,2,\dots,n; \end{matrix} \quad (\text{A-12})$$

then

$$\ln M(\underline{\tau}) \approx \sum_i \tau_i \theta_i + \frac{1}{2} \sum_i \sum_j \tau_i \tau_j \sigma_{ij}. \quad (\text{A-13})$$

Approximations for θ_i^* and τ_i in (A-5) can be derived using (A-13) and taking partial derivatives with respect to the τ_i :

$$\theta_i^* = \frac{\partial \ln M(\underline{\tau})}{\partial \tau_i} \approx \theta_i + \sum_j \sigma_{ij} \tau_j; \quad i=1,2,\dots,n. \quad (\text{A-14})$$

The matrix $\underline{\Sigma} = (\sigma_{ij})$ is the covariance matrix of the $T_i(\omega)$ with respect to the $\pi(\omega)$ distribution.

Relations for $2I(\underline{p}^*:\underline{\pi})$ can now be derived using $\underline{\Sigma}$ and $\underline{\Sigma}^{-1}$ and (A-12), (A-13), and (A-14) in (A-8):

$$\begin{aligned} 2I(\underline{p}^*:\underline{\pi}) &= 2\left(\sum_i \tau_i \theta_i^* - \ln M(\underline{\tau})\right) \\ &\approx 2\left(\sum_i \tau_i (\theta_i + \sum_j \sigma_{ij} \tau_j) - \left(\sum_i \tau_i \theta_i + \frac{1}{2} \sum_i \sum_j \tau_i \tau_j \sigma_{ij}\right)\right) \\ &= 2\left(\sum_i \sum_j \sigma_{ij} \tau_i \tau_j - \frac{1}{2} \sum_i \sum_j \sigma_{ij} \tau_i \tau_j\right) \\ &= \sum_i \sum_j \sigma_{ij} \tau_i \tau_j. \end{aligned}$$

In matrix notation

$$2I(\underline{p}^*:\underline{\pi}) \approx \underline{\tau}' \underline{\Sigma} \underline{\tau}. \quad (\text{A-15})$$

In matrix notation Equation (A-14) can be written

$$\underline{\theta}^* \approx \underline{\theta} + \underline{\Sigma} \underline{\tau}.$$

Solving for $\underline{\tau}$,

$$\underline{\tau} \approx \underline{\Sigma}^{-1}(\underline{\theta}^* - \underline{\theta}).$$

Substituting into (A-15),

$$\begin{aligned} 2I(\underline{p}^* : \underline{\pi}) &\approx (\underline{\theta}^* - \underline{\theta})' \underline{\Sigma}^{-1} \underline{\Sigma} \underline{\Sigma}^{-1} (\underline{\theta}^* - \underline{\theta}) \\ &= (\underline{\theta}^* - \underline{\theta})' \underline{\Sigma}^{-1} (\underline{\theta}^* - \underline{\theta}). \end{aligned} \quad (\text{A-16})$$

Now, the set of $T_i(\omega)$ ($i=1,2,\dots,n$) functions can be partitioned into two sets: one set, H_A , where from (A-2) and (A-11)

$$\theta_i^* = \theta_i; \quad i=1,2,\dots,n_A; \quad (\text{A-17})$$

and the remaining $n-n_A$ $T_i(\omega)$ into the set H_B . The other matrices can be partitioned correspondingly:

$$\underline{\theta}^{*'} = (\underline{\theta}_A^{*'}, \underline{\theta}_B^{*'}), \quad \underline{\theta}' = (\underline{\theta}_A', \underline{\theta}_B'), \quad \underline{\tau}' = (\underline{\tau}_A', \underline{\tau}_B'), \quad (\text{A-18})$$

and the covariance matrix,

$$\underline{\Sigma} = \begin{bmatrix} \underline{\Sigma}_{AA} & \underline{\Sigma}_{AB} \\ \underline{\Sigma}_{BA} & \underline{\Sigma}_{BB} \end{bmatrix}, \quad (\text{A-19})$$

where $\underline{\Sigma}_{AA}$ is $n_A \times n_A$, $\underline{\Sigma}_{BB}$ is $(n-n_A) \times (n-n_A)$, and $(\underline{\Sigma}_{BA})' = \underline{\Sigma}_{AB}$ is $n_A \times (n-n_A)$.

Substituting these relations into (A-14), $\underline{\theta}_i \approx \underline{\theta} + \underline{\tau}$,

$$\begin{aligned}\underline{\theta}_A^* &\approx \underline{\theta}_A + \underline{\tau}_{AA} \underline{\tau}_A + \underline{\tau}_{AB} \underline{\tau}_B \\ \underline{\theta}_B^* &\approx \underline{\theta}_B + \underline{\tau}_{BB} \underline{\tau}_B + \underline{\tau}_{BA} \underline{\tau}_A.\end{aligned}\tag{A-20}$$

But the partition (A-18) was established based on (A-17), so

$\underline{\theta}_A^* = \underline{\theta}_A$, and the approximations (A-15) and (A-16) can be written in terms of the partition. From (A-20)

$$\underline{\tau}_{AA} \underline{\tau}_A + \underline{\tau}_{AB} \underline{\tau}_B \approx \underline{\theta}_A^* - \underline{\theta}_A = 0;$$

therefore,

$$\underline{\tau}_A \approx -\underline{\tau}_{AA}^{-1} \underline{\tau}_{AB} \underline{\tau}_B.$$

From (A-15) and the partition,

$$\begin{aligned}2I(\underline{p}^*; \underline{\pi}) &\approx (\underline{\tau}_A', \underline{\tau}_B') \begin{bmatrix} \underline{\tau}_{AA} & \underline{\tau}_{AB} \\ \underline{\tau}_{BA} & \underline{\tau}_{BB} \end{bmatrix} \begin{bmatrix} \underline{\tau}_A \\ \underline{\tau}_B \end{bmatrix} \\ &= \underline{\tau}_A' \underline{\tau}_{AA} \underline{\tau}_A + \underline{\tau}_B' \underline{\tau}_{BA} \underline{\tau}_A + \underline{\tau}_A' \underline{\tau}_{AB} \underline{\tau}_B + \underline{\tau}_B' \underline{\tau}_{BB} \underline{\tau}_B \\ &= \underline{\tau}_A' (\underline{\tau}_{AA} \underline{\tau}_A + \underline{\tau}_{AB} \underline{\tau}_B) + \underline{\tau}_B' (\underline{\tau}_{BA} \underline{\tau}_A + \underline{\tau}_{BB} \underline{\tau}_B) \\ &= \underline{\tau}_A' (0) + \underline{\tau}_B' (\underline{\tau}_{BB} \underline{\tau}_B - \underline{\tau}_{BA} \underline{\tau}_{AA}^{-1} \underline{\tau}_{AB} \underline{\tau}_B)\end{aligned}$$

$$= \underline{\tau}_B' (\underline{\sum}_{BB} - \underline{\sum}_{BA} \underline{\sum}_{AA}^{-1} \underline{\sum}_{AB}) \underline{\tau}_B.$$

Therefore,

$$2I(p^*:\pi) \approx \underline{\tau}_B' \underline{\sum}_{BB.A} \underline{\tau}_B, \quad (A-21)$$

where

$$\underline{\sum}_{BB.A} = \underline{\sum}_{BB} - \underline{\sum}_{BA} \underline{\sum}_{AA}^{-1} \underline{\sum}_{AB}.$$

From (A-20)

$$\underline{\sum}_{BB} \underline{\tau}_B + \underline{\sum}_{BA} \underline{\tau}_A \approx \underline{\theta}_B^* - \underline{\theta}_B.$$

Substituting for $\underline{\tau}_A$,

$$\underline{\sum}_{BB} \underline{\tau}_B - \underline{\sum}_{BA} \underline{\sum}_{AA}^{-1} \underline{\sum}_{AB} \underline{\tau}_B \approx \underline{\theta}_B^* - \underline{\theta}_B$$

$$\underline{\sum}_{BB.A} \underline{\tau}_B \approx \underline{\theta}_B^* - \underline{\theta}_B$$

$$\underline{\tau}_B \approx \underline{\sum}_{BB.A}^{-1} (\underline{\theta}_B^* - \underline{\theta}_B).$$

Substituting into (A-21),

$$\begin{aligned}
 2I(\underline{p}^* : \underline{\pi}) &\approx (\underline{\theta}_B^* - \underline{\theta}_B)' \sum_{\underline{B}B.A}^{-1} \sum_{\underline{B}B.A} \sum_{\underline{B}B.A}^{-1} (\underline{\theta}_B^* - \underline{\theta}_B) \\
 &= (\underline{\theta}_B^* - \underline{\theta}_B)' \sum_{\underline{B}B.A}^{-1} (\underline{\theta}_B^* - \underline{\theta}_B). \quad (A-22)
 \end{aligned}$$

APPENDIX B

PROBABILITY DESIGNS

TABLE: 2×2

P_{11}	P_{12}	$P_{1\cdot}$
P_{21}	P_{22}	$P_{2\cdot}$
$P_{\cdot 1}$	$P_{\cdot 2}$	1

NO.	$P_{1\cdot}$	$P_{\cdot 1}$	$(P_{11}, P_{12}, P_{21}, P_{22})$
1	.1	.1	(.01, .09, .09, .81)
2	.1	.2	(.02, .08, .18, .72)
3	.1	.3	(.03, .07, .27, .63)
4	.1	.4	(.04, .06, .36, .54)
5	.1	.5	(.05, .05, .45, .45)
6	.2	.2	(.04, .16, .16, .64)
7	.2	.3	(.06, .14, .24, .56)
8	.2	.4	(.08, .12, .32, .48)
9	.2	.5	(.10, .10, .40, .40)
10	.3	.3	(.09, .21, .21, .49)
11	.3	.4	(.12, .18, .28, .42)
12	.3	.5	(.15, .15, .35, .35)
13	.4	.4	(.16, .24, .24, .36)
14	.4	.5	(.20, .20, .30, .30)
15	.5	.5	(.25, .25, .25, .25)

TABLE: 2×3

P_{11}	P_{12}	P_{13}	$P_{1\cdot}$
P_{21}	P_{22}	P_{23}	$P_{2\cdot}$
$P_{\cdot 1}$	$P_{\cdot 2}$	$P_{\cdot 3}$	1

NO.	$P_{1\cdot}$	$P_{\cdot 1}$	$P_{\cdot 2}$	$(P_{11}, P_{12}, P_{13}, P_{21}, P_{22}, P_{23})$
1	.1	.1	.1	(.01, .01, .08, .09, .09, .72)
2	.1	.1	.3	(.01, .03, .06, .09, .27, .54)
3	.1	.2	.2	(.02, .02, .06, .18, .18, .54)
4	.1	.2	.4	(.02, .04, .04, .18, .36, .36)
5	.2	.1	.2	(.02, .04, .14, .08, .16, .56)
6	.2	.1	.4	(.02, .08, .10, .08, .32, .40)
7	.2	.2	.3	(.04, .06, .10, .16, .24, .40)
8	.2	.3	.3	(.06, .06, .08, .24, .24, .32)
9	.3	.1	.1	(.03, .03, .24, .07, .07, .56)
10	.3	.1	.3	(.03, .09, .18, .07, .21, .42)
11	.3	.2	.2	(.06, .06, .18, .14, .14, .42)
12	.3	.2	.4	(.06, .12, .12, .14, .28, .28)
13	.4	.1	.2	(.04, .08, .28, .06, .12, .42)
14	.4	.1	.4	(.04, .16, .20, .06, .24, .30)
15	.4	.2	.3	(.08, .12, .20, .12, .18, .30)
16	.4	.3	.3	(.12, .12, .16, .18, .18, .24)
17	.5	.1	.1	(.05, .05, .40, .05, .05, .40)
18	.5	.1	.3	(.05, .15, .30, .05, .15, .30)
19	.5	.2	.2	(.10, .10, .30, .10, .10, .30)
20	.5	.2	.4	(.10, .20, .20, .10, .20, .20)
21	.5	1/3	1/3	(1/6, 1/6, 1/6, 1/6, 1/6, 1/6)

TABLE: 2 x 4

P ₁₁	P ₁₂	P ₁₃	P ₁₄	P _{1.}
P ₂₁	P ₂₂	P ₂₃	P ₂₄	P _{2.}
P. ₁	P. ₂	P. ₃	P. ₄	1

NO.	<u>P_{1.}</u>	<u>P.₁</u>	<u>P.₂</u>	<u>P.₃</u>	<u>(P₁₁, P₁₂, P₁₃, P₁₄, P₂₁, P₂₂, P₂₃, P₂₄)</u>
1	.1	.1	.1	.1	(.01, .01, .01, .07, .09, .09, .09, .63)
2	.1	.1	.1	.3	(.01, .01, .03, .05, .09, .09, .27, .45)
3	.1	.1	.2	.2	(.01, .02, .02, .05, .09, .18, .18, .45)
4	.1	.2	.2	.2	(.02, .02, .02, .04, .18, .18, .18, .36)
5	.2	.1	.1	.2	(.02, .02, .04, .12, .08, .08, .16, .48)
6	.2	.1	.1	.4	(.02, .02, .08, .08, .08, .08, .32, .32)
7	.2	.1	.2	.3	(.02, .04, .06, .08, .08, .16, .24, .32)
8	.3	.2	.2	.3	(.04, .04, .06, .06, .16, .16, .24, .24)
9	.3	.1	.1	.1	(.03, .03, .03, .21, .07, .07, .07, .49)
10	.3	.1	.1	.3	(.03, .03, .09, .15, .07, .07, .21, .35)
11	.3	.1	.2	.2	(.03, .06, .06, .15, .07, .14, .14, .35)
12	.3	.2	.2	.2	(.06, .06, .06, .12, .14, .14, .14, .28)
13	.4	.1	.1	.2	(.04, .04, .08, .24, .06, .06, .12, .36)
14	.4	.1	.1	.4	(.04, .04, .16, .16, .06, .06, .24, .24)
15	.4	.1	.2	.3	(.04, .08, .12, .16, .06, .12, .18, .24)
16	.4	.2	.2	.3	(.08, .08, .12, .12, .12, .12, .18, .18)
17	.5	.1	.1	.1	(.05, .05, .05, .35, .05, .05, .05, .35)
18	.5	.1	.1	.3	(.05, .05, .15, .25, .05, .05, .15, .25)
19	.5	.1	.2	.2	(.05, .10, .10, .25, .05, .10, .10, .25)
20	.5	.2	.2	.2	(.10, .10, .10, .20, .10, .10, .10, .20)
21	.5	1/4	1/4	1/4	(1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8)

TABLE: 3×3

P_{11}	P_{12}	P_{13}	P_1
P_{21}	P_{22}	P_{23}	P_2
P_{31}	P_{32}	P_{33}	P_3
$P_{\cdot 1}$	$P_{\cdot 2}$	$P_{\cdot 3}$	1

NO.	P_1	P_2	$P_{\cdot 1}$	$P_{\cdot 2}$	$(P_{11}, P_{12}, P_{13}, P_{21}, P_{22}, P_{23}, P_{31}, P_{32}, P_{33})$
1	.1	.1	.1	.1	(.01, .01, .08, .01, .01, .08, .08, .08, .64)
2	.1	.1	.1	.3	(.01, .03, .06, .01, .03, .06, .08, .24, .48)
3	.1	.1	.2	.2	(.02, .02, .06, .02, .02, .06, .16, .16, .48)
4	.1	.1	.2	.4	(.02, .04, .04, .02, .04, .04, .16, .32, .32)
5	.1	.2	.1	.2	(.01, .02, .07, .02, .04, .14, .07, .14, .49)
6	.1	.2	.1	.4	(.01, .04, .05, .02, .08, .10, .07, .28, .35)
7	.1	.2	.2	.3	(.02, .03, .05, .04, .06, .10, .14, .2, .35)
8	.1	.2	.3	.3	(.03, .03, .04, .06, .06, .08, .21, .21, .28)
9	.1	.3	.1	.3	(.01, .03, .06, .03, .09, .18, .06, .18, .36)
10	.1	.3	.2	.2	(.02, .02, .06, .06, .06, .18, .12, .12, .36)
11	.1	.3	.2	.4	(.02, .04, .04, .06, .12, .12, .12, .24, .24)
12	.1	.4	.1	.4	(.01, .04, .05, .04, .16, .20, .05, .20, .25)
13	.1	.4	.2	.3	(.02, .03, .05, .08, .12, .20, .10, .15, .25)
14	.1	.4	.3	.3	(.03, .03, .04, .12, .12, .16, .15, .15, .20)
15	.2	.2	.2	.2	(.04, .04, .12, .04, .04, .12, .12, .12, .36)
16	.2	.2	.2	.4	(.04, .08, .08, .04, .08, .08, .12, .24, .24)
17	.2	.3	.2	.3	(.04, .06, .10, .06, .09, .15, .10, .15, .25)
18	.2	.3	.3	.3	(.06, .06, .08, .09, .09, .12, .15, .15, .20)
19	.2	.4	.2	.4	(.04, .08, .08, .08, .16, .16, .08, .16, .16)
20	.3	.3	.3	.3	(.09, .09, .12, .09, .09, .12, .12, .12, .16)
21	1/3	1/3	1/3	1/3	(1/9, 1/9, 1/9, 1/9, 1/9, 1/9, 1/9, 1/9, 1/9)

TABLE: 2 x 5

P ₁₁	P ₁₂	P ₁₃	P ₁₄	P ₁₅	P _{1.}
P ₂₁	P ₂₂	P ₂₃	P ₂₄	P ₂₅	P _{2.}
P. ₁	P. ₂	P. ₃	P. ₄	P. ₅	1

NO.	P _{1.}	P. ₁	P. ₂	P. ₃	P. ₄	(P ₁₁ , P ₁₂ , P ₁₃ , P ₁₄ , P ₁₅ , P ₂₁ , P ₂₂ , P ₂₃ , P ₂₄ , P ₂₅)
1	.1	.1	.1	.1	.1	(.01,.01,.01,.01,.06,.09,.09,.09,.09,.54)
2	.1	.1	.1	.1	.3	(.01,.01,.01,.03,.04,.09,.09,.09,.27,.36)
3	.1	.1	.1	.2	.3	(.01,.01,.02,.03,.03,.09,.09,.18,.27,.27)
4	.2	.1	.1	.1	.2	(.02,.02,.02,.04,.10,.08,.08,.08,.16,.40)
5	.2	.1	.1	.2	.2	(.02,.02,.04,.04,.08,.08,.08,.16,.16,.32)
6	.2	.1	.2	.2	.2	(.02,.04,.04,.04,.06,.08,.16,.16,.16,.24)
7	.3	.1	.1	.1	.1	(.03,.03,.03,.03,.18,.07,.07,.07,.07,.42)
8	.3	.1	.1	.1	.3	(.03,.03,.03,.09,.12,.07,.07,.07,.21,.28)
9	.3	.1	.1	.2	.3	(.03,.03,.06,.09,.09,.07,.07,.14,.21,.21)
10	.4	.1	.1	.1	.2	(.04,.04,.04,.08,.20,.06,.06,.06,.12,.30)
11	.4	.1	.1	.2	.2	(.04,.04,.08,.08,.16,.06,.06,.12,.12,.24)
12	.1	.2	.2	.2	.2	(.04,.08,.08,.08,.12,.06,.12,.12,.12,.18)
13	.5	.1	.1	.1	.1	(.05,.05,.05,.05,.30,.05,.05,.05,.05,.30)
14	.5	.1	.1	.1	.3	(.05,.05,.05,.15,.20,.05,.05,.05,.15,.20)
15	.5	.1	.1	.2	.3	(.05,.05,.10,.15,.15,.05,.05,.10,.15,.15)
16	.5	.2	.2	.2	.2	(.10,.10,.10,.10,.10,.10,.10,.10,.10,.10)

TABLE : $2 \times 2 \times 2$

P_{111}	P_{112}	$P_{11\cdot}$	P_{211}	P_{212}	$P_{21\cdot}$
P_{121}	P_{122}	$P_{12\cdot}$	P_{221}	P_{222}	$P_{22\cdot}$
$P_{1\cdot 1}$	$P_{1\cdot 2}$	$P_{1\cdot\cdot}$	$P_{2\cdot 1}$	$P_{2\cdot 2}$	$P_{2\cdot\cdot}$

$$P_{\cdot\cdot 1} = P_{111} + P_{121} + P_{211} + P_{221}$$

$$P_{\cdot\cdot 2} = P_{112} + P_{122} + P_{212} + P_{222}$$

NO.	$P_{1\cdot\cdot}$	$P_{\cdot 1\cdot}$	$P_{\cdot\cdot 1}$	$(P_{111}, P_{112}, P_{121}, P_{122}, P_{211}, P_{212}, P_{221}, P_{222})$
*1	.1	.1	.1	(.001, .009, .009, .081, .009, .081, .081, .729)
*2	.1	.1	.3	(.003, .007, .027, .063, .027, .063, .243, .567)
*3	.1	.1	.5	(.005, .005, .045, .045, .045, .045, .405, .405)
*4	.1	.2	.2	(.004, .016, .016, .064, .036, .144, .144, .576)
5	.1	.2	.4	(.008, .012, .032, .048, .072, .108, .288, .432)
6	.1	.3	.3	(.009, .021, .021, .049, .081, .189, .189, .441)
7	.1	.3	.5	(.015, .015, .035, .035, .135, .135, .315, .315)
8	.1	.4	.4	(.016, .024, .024, .036, .144, .216, .216, .324)
9	.1	.5	.5	(.025, .025, .025, .025, .225, .225, .225, .225)
*10	.2	.2	.2	(.008, .032, .032, .128, .032, .128, .128, .512)
11	.2	.2	.4	(.016, .024, .064, .096, .064, .096, .256, .384)
12	.2	.3	.3	(.018, .042, .042, .098, .072, .168, .168, .392)
13	.2	.3	.5	(.030, .030, .070, .070, .120, .120, .280, .280)
14	.2	.4	.4	(.032, .048, .048, .072, .128, .192, .192, .288)
15	.2	.5	.5	(.050, .050, .050, .050, .200, .200, .200, .200)
16	.3	.3	.3	(.027, .063, .063, .147, .063, .147, .147, .343)
17	.3	.3	.5	(.045, .045, .105, .105, .105, .105, .245, .245)
18	.3	.4	.4	(.048, .072, .072, .108, .112, .168, .168, .252)
19	.3	.5	.5	(.075, .075, .075, .075, .175, .175, .175, .175)
20	.4	.4	.4	(.064, .096, .096, .144, .096, .144, .144, .216)
21	.4	.5	.5	(.100, .100, .100, .100, .150, .150, .150, .150)
22	.5	.5	.5	(.125, .125, .125, .125, .125, .125, .125, .125)

* Not used for "no second-order" interaction hypothesis

TABLE: $2 \times 2 \times 3$

P_{111}	P_{112}	P_{113}	$P_{11\cdot}$	P_{211}	P_{212}	P_{213}	$P_{21\cdot}$
P_{121}	P_{122}	P_{123}	$P_{12\cdot}$	P_{221}	P_{222}	P_{223}	$P_{22\cdot}$
$P_{1\cdot 1}$	$P_{1\cdot 2}$	$P_{1\cdot 3}$	$P_{1\cdot\cdot}$	$P_{2\cdot 1}$	$P_{2\cdot 2}$	$P_{2\cdot 3}$	$P_{2\cdot\cdot}$

$$P_{\cdot\cdot 1} = P_{111} + P_{121} + P_{211} + P_{221}$$

$$P_{\cdot\cdot 2} = P_{112} + P_{122} + P_{212} + P_{222}$$

$$P_{\cdot\cdot 3} = P_{113} + P_{123} + P_{213} + P_{223}$$

NO.	$P_{1\cdot\cdot}$	$P_{\cdot 1\cdot}$	$P_{\cdot\cdot 1}$	$P_{\cdot\cdot 2}$	$(P_{111}, P_{112}, P_{113}, P_{121}, P_{122}, P_{123},$ $P_{211}, P_{212}, P_{213}, P_{221}, P_{222}, P_{223})$
1	.1	.1	.1	.1	(.001,.001,.008,.009,.009,.072, .009,.009,.072,.081,.081,.648)
2	.1	.1	.2	.2	(.002,.002,.006,.018,.018,.054, .018,.018,.054,.162,.162,.486)
3	.1	.2	.1	.2	(.002,.004,.014,.008,.016,.056, .018,.036,.126,.072,.144,.504)
4	.1	.2	.2	.3	(.004,.006,.010,.016,.024,.040, .036,.054,.090,.144,.216,.360)
5	.1	.3	.1	.3	(.003,.009,.018,.007,.021,.042, .027,.081,.162,.063,.189,.378)
6	.1	.3	.2	.4	(.006,.012,.012,.014,.028,.028, .054,.108,.108,.126,.252,.252)
7	.1	.4	.1	.4	(.004,.016,.020,.006,.024,.030, .036,.144,.180,.054,.216,.270)
8	.1	.4	.3	.3	(.012,.012,.016,.018,.018,.024, .10-.108,.144,.162,.162,.216)
9	.1	.5	.1	.1	(.005,.005,.040,.005,.005,.040, .045,.045,.360,.045,.045,.360)
10	.1	.5	.2	.2	(.010,.010,.030,.010,.010,.030, .090,.090,.270,.090,.090,.270)
11	.2	.2	.1	.2	(.004,.008,.028,.016,.032,.112, .016,.032,.112,.064,.128,.448)
12	.2	.2	.2	.3	(.008,.012,.020,.032,.048,.080, .032,.048,.080,.128,.192,.320)
13	.2	.3	.1	.3	(.006,.018,.036,.014,.042,.084, .024,.072,.144,.056,.168,.336)
14	.2	.3	.2	.4	(.012,.024,.024,.028,.056,.056, .048,.096,.096,.112,.224,.224)
15	.2	.4	.1	.4	(.008,.032,.040,.012,.048,.060, .032,.128,.160,.048,.192,.240)
16	.2	.4	.3	.3	(.024,.024,.032,.036,.036,.048, .096,.096,.128,.144,.144,.192)
17	.2	.5	.1	.1	(.010,.010,.080,.010,.010,.080, .040,.040,.320,.040,.040,.320)
18	.2	.5	.2	.2	(.020,.020,.060,.020,.020,.060, .080,.080,.240,.080,.080,.240)

TABLE: $2 \times 2 \times 3$ continued

NO.	$P_{1..}$	$P_{.1.}$	$P_{..1}$	$P_{..2}$	$(P_{111}, P_{112}, P_{113}, P_{121}, P_{122}, P_{123},$ $P_{211}, P_{212}, P_{213}, P_{221}, P_{222}, P_{223})$
19	.3	.3	.1	.2	(.009, .018, .063, .021, .042, .147, .021, .042, .147, .049, .098, .343)
20	.3	.3	.2	.3	(.018, .027, .045, .042, .063, .105, .042, .063, .105, .098, .147, .245)
21	.3	.4	.1	.3	(.012, .036, .072, .018, .054, .108, .028, .084, .168, .042, .126, .252)
22	.3	.4	.2	.4	(.024, .048, .048, .036, .072, .072, .056, .112, .112, .084, .168, .168)
23	.3	.5	.1	.4	(.015, .060, .075, .015, .060, .075, .035, .140, .175, .035, .140, .175)
24	.3	.5	.3	.3	(.045, .045, .060, .045, .045, .060, .105, .105, .140, .105, .105, .140)
25	.4	.4	.1	.1	(.016, .016, .128, .024, .024, .192, .024, .024, .192, .036, .036, .288)
26	.4	.4	.2	.2	(.032, .032, .096, .048, .048, .144, .048, .048, .144, .072, .072, .216)
27	.4	.5	.1	.2	(.020, .040, .140, .020, .040, .140, .030, .060, .210, .030, .060, .210)
28	.4	.5	.2	.3	(.040, .060, .100, .040, .060, .100, .060, .090, .150, .060, .090, .150)
29	.5	.5	.1	.3	(.025, .075, .150, .025, .075, .150, .025, .075, .150, .025, .075, .150)
30	.5	.5	.2	.4	(.050, .100, .100, .050, .100, .100, .050, .100, .100, .050, .100, .100)
31	.5	.5	1/3	1/3	(1/12, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12)

APPENDIX C

SAMPLE SIZE DESIGN

TABLE	k #CELLS	N SAMPLE SIZES	\bar{m} AVE. EXPECTED VALUES	TOTAL #N's
E2 × 2	4	4(4)36	1(1)9	9
		40(8)56	10(2)14	3
M2 × 2	4	64(8)96	16(2)24	5
2 × 3	6	12(6)30	2(1)5	4
2 × 4	8	16(8)40	2(1)5	4
		48(16)96	6(2)12	4
3 × 3	9	18(9)45	2(1)5	4
		54(18)108	6(2)12	4
2 × 5	10	20(10)100	2(1)10	9
2 × 2 × 2	8	16(8)40	2(1)5	4
		48(16)96	6(2)12	4
2 × 2 × 3	12	24(12)96	2(1)8	7
S2 × 2 × 2	8	24(16(104)	3(2)13	6

E - Exact

M - Monte Carlo

S - No Second-Order Interaction

APPENDIX D

EXACT 2×2 PROGRAM

```

C**** THIS PROGRAM CALCULATES EXACT LEVELS OF SIGNIFICANCE (BASED
C      ON NOMINAL LEVELS .10,.05,.01) FOR THREE ASYMPTOTIC
C      CHI-SQUARE STATISTICS (PEARSON,KULLBACK,GSK) WHEN USED AS
C      INDEPENDENCE HYPOTHESIS TEST STATISTICS IN 2X2 CONTINGENCY
C**** TABLES WITH SAMPLE SIZE N (DIVISIBLE BY FOUR).
      PROGRAM EXCTA (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
      DIMENSION A(3,3)
      COMMON X(4),P(4),XN
      COMMON/STAT/G2,X2,X2GSK1
      COMMON/HYP/FTOTAL,FSTAT(3,3),F
      WRITE(6,15)
15  FORMAT(50X,'EXACT ALPHAS FOR CTA STATISTICS',/)
      WRITE(6,25)
25  FORMAT(31X,'NOMINAL ALPHA = .10',9X,'NOMINAL ALPHA = .05',
      +9X,'NOMINAL ALPHA = .01',/)
      WRITE(6,35)
35  FORMAT(5X,'P VECTOR',10X,'N',5X,'KULLBACK',1X,'PEARSON',
      +3X,'GSK1',2(5X,'KULLBACK',1X,'PEARSON',3X,'GSK1'),4X,'FTOTAL')
C**** CYCLE THROUGH PROBABILITY VECTORS OF DESIGN.
      DO 50 L=1,5
      PIDOT=FLOAT(L)*.1
      DO 40 M=L,5
      PDOT1=FLOAT(M)*.1
      P(1)=PIDOT*PDOT1
      P(2)=PIDOT*(1.-PDOT1)
      P(3)=(1.-PIDOT)*PDOT1
      P(4)=(1.-PIDOT)*(1.-PDOT1)
C**** ENTER SAMPLE SIZES IN DO LOOP (INCREMENTED BY FOUR).
      DO 30 N=4,40,4
      XN=N
C**** INITIALIZE PARAMETERS: FTOTAL - TOTAL MULTINOMIAL PROBABILITY,
C**** FSTAT(I,J) - ACCUMULATED MULTINOMIAL REJECTION PROBABILITIES.
      FTOTAL=0.0
      DO 20 I=1,3
      DO 10 J=1,3
      FSTAT(I,J)=0.0
10  CONTINUE
20  CONTINUE
C**** ACCUMULATE FSTAT FOR GROUP E1 (ALL CELLS EQUAL).
      DO 200 I=1,4
      X(I)=XN/4.0

```

```

200 CONTINUE
   F=FMULT()
   CALL STATS
   CALL HYPACC
C**** ACCUMULATE FSTAT FOR GROUP E2 (BOTH CORNERS EQUAL).
   DO 300 I=0,(N/4-1)
     X(1)=FLOAT(I)
     X(4)=X(1)
     X(2)=(XN-(2.*X(1)))/2.
     X(3)=X(2)
     F=2.0*FMULT()
     CALL STATS
     CALL HYPACC
300 CONTINUE
   IF(N.EQ.4)GO TO 410
C**** ACCUMULATE FSTAT FOR GROUP E3 (BOTH SIDES EQUAL).
   DO 400 I=1,(N/4-1)
     X(1)=FLOAT(I)
     X(2)=X(1)
     X(3)=(XN-(2.*X(1)))/2.0
     X(4)=X(3)
     F=FMULT()
     CALL ROT4(F)
     CALL STATS
     CALL HYPACC
400 CONTINUE
410 DO 500 I=1,(N/2-1)
     X(1)=0.0
     X(4)=0.0
     X(2)=FLOAT(I)
     X(3)=XN-X(2)
     F=FMULT()
     CALL ROT4(F)
     CALL STATS
     CALL HYPACC
500 CONTINUE
C**** ACCUMULATE FSTAT FOR GROUP E4 (ONE SET EQUAL CORNERS).
   DO 550 I=1,(N/2-1)
     DO 540 J=0,((N-2*I-2)/2)
       X(1)=FLOAT(I)
       X(4)=X(1)
       X(2)=FLOAT(J)
       X(3)=XN-X(1)-X(2)-X(4)
       F=FMULT()
       CALL ROT4(F)
       CALL STATS
       CALL HYPACC
540 CONTINUE
550 CONTINUE
C**** ACCUMULATE FSTAT FOR GROUP O (ALL OTHERS).
   DO 610 I=1,((N-2)/2)
     DO 600 J=(I+1),(N-I-1)
       X(1)=0.0

```

```

X(2)=FLOAT(I)
X(3)=FLOAT(J)
X(4)=XN-X(2)-X(3)
F=FMULT()
CALL ROT8(F)
CALL STATS
CALL HYPACC
600 CONTINUE
610 CONTINUE
IF(N.EQ.4)GO TO 700
DO 660 I=1,((N-4)/4)
DO 650 J=(I+1),((N-2*I-2)/2)
X(1)=FLOAT(I)
X(2)=X(1)
X(3)=FLOAT(J)
X(4)=XN-X(1)-X(2)-X(3)
F=FMULT()
CALL ROT8(F)
CALL STATS
CALL HYPACC
650 CONTINUE
660 CONTINUE
DO 690 I=1,((N-4)/4)
DO 680 J=(I+1),((N-2*I-2)/2)
DO 670 K=(J+1),(N-2*I-J-1)
X(1)=FLOAT(I)
X(2)=FLOAT(J)
X(3)=FLOAT(K)
X(4)=XN-X(1)-X(2)-X(3)
F=FMULT()
CALL ROT8(F)
CALL STATS
CALL HYPACC
670 CONTINUE
680 CONTINUE
690 CONTINUE
C**** CALCULATE EXACT LEVEL OF SIGNIFICANCE A(I,J) FOR EACH
C**** STATISTIC(I) AND NOMINAL LEVEL(J) COMBINATION AND PRINT RESULTS.
700 DO 720 I=1,3
DO 710 J=1,3
A(I,J)=FSTAT(I,J)/FTOTAL
710 CONTINUE
720 CONTINUE
WRITE(6,45)(P(I),I=1,4),N,((A(I,J),I=1,3),J=1,3),FTOTAL
45 FORMAT(2H (,3(F3.2,','),F3.2,')',3X,I3,5X,F6.4,
+2(3X,F6.4),1X,3(3X,F6.4),1X,3(3X,F6.4),2X,F6.4)
30 CONTINUE
40 CONTINUE
50 CONTINUE
STOP
END
C**** FMULT - FUNCTION TO CALCULATE MULTINOMIAL PROBABILITIES.
FUNCTION FMULT()

```

```

COMMON X(4),P(4),XN
FMULT=1.0
DO 100 I=1,4
FMULT=FMULT*(P(I)**X(I))/FACT(X(I))
100 CONTINUE
FMULT=FMULT*FACT(XN)
RETURN
END
C**** FACT - FUNCTION TO CALCULATE FACTORIALS.
FUNCTION FACT(Z)
FACT=1.0
IF(Z.EQ.0.0)GO TO 125
DO 120 I=2,IFIX(Z)
FACT=FACT*FLOAT(I)
120 CONTINUE
125 RETURN
END
C**** HYPACC - SUBROUTINE TO PERFORM ALL HYPOTHESIS TESTS AND
C**** ACCUMULATE MULTINOMIAL REJECTION PROBABILITIES.
SUBROUTINE HYPACC
COMMON/HYP/FTOTAL,FSTAT(3,3),F
COMMON/STAT/G2,X2,X2GSK1
FTOTAL=FTOTAL+F
IF(G2.GT.6.63490)GO TO 101
IF(G2.GT.3.84146)GO TO 102
IF(G2.GT.2.70554)GO TO 103
GO TO 104
101 FSTAT(1,3)=FSTAT(1,3)+F
102 FSTAT(1,2)=FSTAT(1,2)+F
103 FSTAT(1,1)=FSTAT(1,1)+F
104 IF(X2.GT.6.63490)GO TO 106
IF(X2.GT.3.84146)GO TO 107
IF(X2.GT.2.70554)GO TO 108
GO TO 109
106 FSTAT(2,3)=FSTAT(2,3)+F
107 FSTAT(2,2)=FSTAT(2,2)+F
108 FSTAT(2,1)=FSTAT(2,1)+F
109 IF(X2GSK1.GT.6.63490)GO TO 111
IF(X2GSK1.GT.3.84146)GO TO 112
IF(X2GSK1.GT.2.70554)GO TO 113
GO TO 114
111 FSTAT(3,3)=FSTAT(3,3)+F
112 FSTAT(3,2)=FSTAT(3,2)+F
113 FSTAT(3,1)=FSTAT(3,1)+F
114 RETURN
END
C**** STATS - SUBROUTINE TO CALCULATE VALUES FOR THREE CHI-SQUARE
C**** STATISTICS: PEARSON(X2), KULLBACK(G2), AND GSK(X2GSK1).
SUBROUTINE STATS
DIMENSION XM(4)
COMMON X(4),P(4),XN
COMMON/STAT/G2,X2,X2GSK1
X1DOT=X(1)+X(2)

```

```

XDOT1=X(1)+X(3)
XM(1)=X1DOT*XDOT1/XN
XM(2)=X1DOT*(XN-XDOT1)/XN
XM(3)=(XN-X1DOT)*XDOT1/XN
XM(4)=(XN-X1DOT)*(XN-XDOT1)/XN
X2=((X(1)-XM(1))**2)*(1./XM(1)+1./XM(2)+1./XM(3)+1./XM(4))
G2=0.0
DO 110 I=1,4
IF(X(I).EQ.0.0)GO TO 105
G2=G2+2.0*X(I)*LOG(X(I)/XM(I))
GO TO 110
105 X(I)=0.25
110 CONTINUE
X2GSK1=((LOG(X(1))-LOG(X(2))-LOG(X(3))+LOG(X(4)))
+**2)/(1./X(1)+1./X(2)+1./X(3)+1./X(4))
RETURN
END
C**** ROT8 - SUBROUTINE TO ROTATE CONTINGENCY TABLE THROUGH EIGHT
C**** SYMMETRIC POSITIONS.
SUBROUTINE ROT8(F)
COMMON X(4),P(4),XN
JJ=1
130 DO 140 I=1,3
X(1)=X(2)
X(2)=X(4)
X(4)=X(3)
X(3)=XN-X(1)-X(2)-X(4)
F=F+FMULT()
140 CONTINUE
IF(JJ.EQ.0)GO TO 150
A=X(1)
X(1)=X(2)
X(2)=A
X(3)=X(4)
X(4)=XN-X(1)-X(2)-X(3)
F=F+FMULT()
JJ=0
GO TO 130
150 RETURN
END
C**** ROT4 - SUBROUTINE TO ROTATE CONTINGENCY TABLE THROUGH FOUR
C**** SYMMETRIC POSITIONS.
SUBROUTINE ROT4(F)
COMMON X(4),P(4),XN
DO 170 I=1,3
X(1)=X(2)
X(2)=X(4)
X(4)=X(3)
X(3)=XN-X(1)-X(2)-X(4)
F=F+FMULT()
170 CONTINUE
RETURN
END

```


APPENDIX E

MONTE CARLO 2 x 3 PROGRAM

```

C**** THIS PROGRAM CALCULATES ESTIMATES OF EXACT LEVELS OF
C SIGNIFICANCE (BASED ON NOMINAL LEVELS .10,.05,.01) FOR
C THREE ASYMPTOTIC CHI-SQUARE STATISTICS (PEARSON,KULLBACK,
C GSK) WHEN USED AS INDEPENDENT HYPOTHESIS TEST STATISTICS
C IN 2X3 CONTINGENCY TABLES. THE PROCEDURE IS A MONTE CARLO
C SIMULATION USING 2000 RANDOMLY GENERATED TABLES BASED ON
C THE MULTINOMIAL DISTRIBUTION.
C THE PROGRAM IS DIVIDED INTO FOUR PARTS: A MAIN PROGRAM AND
C THREE SUBROUTINES. THE MAIN PROGRAM CONTROLS THE CYCLING
C PROCESS THROUGH THE VARIOUS PROBABILITY DESIGNS AND SAMPLE
C SIZES, CALLS THE SUBROUTINES, MAKES THE FINAL CALCULATIONS
C FOR THE ESTIMATED EXACT LEVELS OF SIGNIFICANCE, AND
C**** PRINTS THE RESULTS. THE SUBROUTINES ARE DESCRIBED BELOW.
PROGRAM MC2BY3(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
DIMENSION A(3,3)
DOUBLE PRECISION DSEED
COMMON X(2000,6),NT,N
COMMON/SG/P(6),DSEED
COMMON/STAT/G2(2000),X2(2000),X2GSK1(2000)
COMMON/SHYP/HI(3,3)
WRITE(6,15)
15 FORMAT(53X,'ESTIMATED EXACT ALPHAS FOR CTA STATISTICS',/)
WRITE(6,25)
25 FORMAT(39X,'NOMINAL ALPHA = .10',9X,'NOMINAL ALPHA = .05',
+9X,'NOMINAL ALPHA = .01',/)
WRITE(6,35)
35 FORMAT(9X,'P VECTOR',14X,'N',5X,'KULLBACK',1X,'PEARSON',
+3X,'GSK1',2(5X,'KULLBACK',1X,'PEARSON',3X,'GSK1'))
NT=2000
XNT=NT
C**** DSEED - RANDOM NUMBER GENERATOR SEED IN RANGE
C**** (1.DO , 2147483647.DO).
DSEED=18473.0D0
C**** IND - INDICATOR FOR EQUIPROBABLE TABLE.
IND=0
C**** CYCLE THROUGH PROBABILITY VECTORS OF DESIGN.
DO 60 I=1,5,2
PIDOT=FLOAT(I)*.1
P2DOT=1-PIDOT
DO 50 J=1,(3-MOD(I,2))
PDOT1=FLOAT(J)*.1

```

```

IF(MOD(I,2).EQ.1)II=J
IF(MOD(I,2).EQ.0)II=(5*J-J**2)/2
IF(MOD(I,2).EQ.1)JJ=J+2
IF(MOD(I,2).EQ.0)JJ=(J**2-5*J)/2+6
DO 40 K=II,JJ,2
PDOT2=FLOAT(K)*.1
PDOT3=1-PDOT1-PDOT2
P(1)=PIDOT*PDOT1
P(2)=PIDOT*PDOT2
P(3)=PIDOT*PDOT3
P(4)=(1.-PIDOT)*PDOT1
P(5)=(1.-PIDOT)*PDOT2
P(6)=(1.-PIDOT)*PDOT3
C**** CYCLE THROUGH SAMPLE SIZES, CALL SUBROUTINES, CALCULATE
C ESTIMATES OF EXACT LEVELS OF SIGNIFICANCE A(L,M) FOR EACH
C STATISTIC(L) AND NOMINAL LEVEL(M) COMBINATION, AND
C**** PRINT RESULTS.
100 DO 30 N=12,30,6
CALL GEN
CALL STATS
CALL HYP
DO 20 L=1,3
DO 10 M=1,3
A(L,M)=HI(L,M)/XNT
10 CONTINUE
20 CONTINUE
WRITE(6,45)(P(L),L=1,6),N,((A(L,M),L=1,3),M=1,3)
45 FORMAT(2H (,5(F3.2,','),F3.2,')',3X,I3,5X,F6.4,
+2(3X,F6.4),1X,3(3X,F6.4),1X,3(3X,F6.4))
30 CONTINUE
IF(IND.EQ.1)GO TO 74
40 CONTINUE
50 CONTINUE
60 CONTINUE
C**** DESIGNATE EQUIPROBABLE TABLE.
DO 70 L=1,6
P(L)=1.0/6.0
70 CONTINUE
IND=1
GO TO 100
74 STOP
END
C**** GEN - SUBROUTINE USED TO GENERATE 2000 RANDOM MULTINOMIAL
C OBSERVATIONS. USES IMSL(1980), SUBROUTINE GGMN CODING
C MODIFIED FOR EFFICIENCY AND SUBROUTINE GGBN. ALL TABLES
C ARE CHECKED FOR USABILITY BASED ON THE "NO ZERO MARGINALS"
C CRITERION. THE PROCEDURE IS CONTINUED UNTIL 2000 USABLE
C**** TABLES HAVE BEEN GENERATED.
SUBROUTINE GEN
DOUBLE PRECISION DSEED
DIMENSION IB(1)
COMMON X(2000,6),NT,N
COMMON/SG/P(6),DSEED

```

```

      DO 90 L=1,NT
75  NUSED=0
      PLEFT=1.0
      DO 80 MM=1,5
      M=7-MM
      PR=P(M)/PLEFT
      PLEFT=PLEFT-P(M)
      NLEFT=N-NUSED
      CALL GGBN(DSEED,1,NLEFT,PR,IB)
      X(L,M)=IB(1)
      NUSED=NUSED+IB(1)
80  CONTINUE
      X(L,1)=N-NUSED
      IF(X(L,1)+X(L,4).EQ.0.0)GO TO 75
      IF(X(L,2)+X(L,5).EQ.0.0)GO TO 75
      IF(X(L,3)+X(L,6).EQ.0.0)GO TO 75
      IF(X(L,1)+X(L,2)+X(L,3).EQ.0.0)GO TO 75
      IF(X(L,4)+X(L,5)+X(L,6).EQ.0.0)GO TO 75
90  CONTINUE
      RETURN
      END
C**** STATS - SUBROUTINE TO CALCULATE THE VALUES OF THREE CHI-SQUARE
C**** STATISTICS PEARSON(X2), KULLBACK(G2), AND GSK(X2GSK1).
      SUBROUTINE STATS
      DIMENSION XM(6),XINV(6)
      COMMON X(2000,6),NT,N
      COMMON/STAT/G2(2000),X2(2000),X2GSK1(2000)
      XN=N
      DO 120 L=1,NT
      X1DOT=X(L,1)+X(L,2)+X(L,3)
      X2DOT=X(L,4)+X(L,5)+X(L,6)
      XDOT1=X(L,1)+X(L,4)
      XDOT2=X(L,2)+X(L,5)
      XDOT3=X(L,3)+X(L,6)
      XM(1)=X1DOT*XDOT1/XN
      XM(2)=X1DOT*XDOT2/XN
      XM(3)=X1DOT*XDOT3/XN
      XM(4)=X2DOT*XDOT1/XN
      XM(5)=X2DOT*XDOT2/XN
      XM(6)=X2DOT*XDOT3/XN
      X2(L)=0.0
      G2(L)=0.0
      DO 110 M=1,6
      X2(L)=X2(L)+((X(L,M)-XM(M))**2)/XM(M)
      IF (X(L,M).EQ.0.0)GO TO 105
      G2(L)=G2(L)+2.0*X(L,M)*LOG(X(L,M)/XM(M))
      GO TO 107
105  X(L,M)=1.0/6.0
107  XINV(M)=1./X(L,M)
110  CONTINUE
      F11=LOG(X(L,1)*X(L,6)/(X(L,3)*X(L,4)))
      F12=LOG(X(L,2)*X(L,6)/(X(L,3)*X(L,5)))
      SUM3=XINV(3)+XINV(6)

```

```

SUM1=XINV(1)+XINV(4)+SUM3
SUM2=XINV(2)+XINV(5)+SUM3
X2GSK1(L)=((F11**2)*SUM2+(F12**2)*SUM1-2.*F11*F12*SUM3)/
+(SUM1*SUM2-SUM3**2)
120 CONTINUE
RETURN
END
C**** HYP - SUBROUTINE TO PERFORM ALL HYPOTHESIS TESTS AND
C ACCUMULATE THE NUMBER OF REJECTIONS HI(L,M) FOR EACH
C**** STATISTIC(L) AND NOMINAL LEVEL(M) COMBINATION.
SUBROUTINE HYP
COMMON X(2000,6),NT,N
COMMON/STAT/G2(2000),X2(2000),X2GSK1(2000)
COMMON/SHYP/HI(3,3)
DO 150 L=1,3
DO 140 M=1,3
HI(L,M)=0.0
140 CONTINUE
150 CONTINUE
DO 440 L=1,NT
IF(G2(L).GT.9.21034)GO TO 151
IF(G2(L).GT.5.99147)GO TO 152
IF(G2(L).GT.4.60517)GO TO 153
GO TO 154
151 HI(1,3)=HI(1,3)+1.0
152 HI(1,2)=HI(1,2)+1.0
153 HI(1,1)=HI(1,1)+1.0
154 IF(X2(L).GT.9.21034)GO TO 156
IF(X2(L).GT.5.99147)GO TO 157
IF(X2(L).GT.4.60517)GO TO 158
GO TO 159
156 HI(2,3)=HI(2,3)+1.0
157 HI(2,2)=HI(2,2)+1.0
158 HI(2,1)=HI(2,1)+1.0
159 IF(X2GSK1(L).GT.9.21034)GO TO 161
IF(X2GSK1(L).GT.5.99147)GO TO 162
IF(X2GSK1(L).GT.4.60517)GO TO 163
GO TO 164
161 HI(3,3)=HI(3,3)+1.0
162 HI(3,2)=HI(3,2)+1.0
163 HI(3,1)=HI(3,1)+1.0
164 CONTINUE
440 CONTINUE
RETURN
END

```

APPENDIX F

EXACT 2×2 DATA

EXACT ALPHAS FOR CTA STATISTICS

P VECTOR	N	NOMINAL ALPHA = .10			NOMINAL ALPHA = .05			NOMINAL ALPHA = .01		
		KULLBACK	PEARSON	GSKI	KULLBACK	PEARSON	GSKI	KULLBACK	PEARSON	GSKI
(.01-.09-.09-.81)	4	.1910	.1910	0.0000	.1910	.1910	0.0000	0.0000	0.0000	0.0000
(.01-.09-.09-.81)	8	.1526	.1507	.0025	.0630	.0624	0.0000	.0025	.0589	0.0000
(.01-.09-.09-.81)	12	.1220	.1200	.0300	.0870	.0862	.0016	.0282	.0282	0.0000
(.01-.09-.09-.81)	16	.0964	.1231	.0219	.0544	.0805	.0064	.0145	.0507	0.0000
(.01-.09-.09-.81)	20	.0798	.1004	.0377	.0506	.0909	.0056	.0102	.0337	.0000
(.01-.09-.09-.81)	24	.0821	.1134	.0450	.0412	.0747	.0112	.0083	.0353	.0002
(.01-.09-.09-.81)	28	.0660	.0948	.0496	.0411	.0602	.0109	.0082	.0280	.0002
(.01-.09-.09-.81)	32	.0617	.0909	.0597	.0385	.0736	.0152	.0069	.0319	.0006
(.01-.09-.09-.81)	36	.0572	.0834	.0546	.0323	.0590	.0222	.0062	.0278	.0019
(.01-.09-.09-.81)	40	.0616	.0956	.0491	.0367	.0551	.0230	.0053	.0264	.0018
(.01-.09-.09-.81)	48	.0574	.0779	.0565	.0275	.0268	.0268	.0039	.0258	.0031
(.01-.09-.09-.81)	56	.0602	.0744	.0567	.0271	.0476	.0273	.0037	.0215	.0040
(.02-.08-.18-.72)	4	.1710	.1710	0.0000	.1710	.1710	0.0000	0.0000	0.0000	0.0000
(.02-.08-.18-.72)	8	.1407	.1333	.0035	.0475	.0455	0.0000	.0035	.0374	0.0000
(.02-.08-.18-.72)	12	.1198	.1110	.0216	.0670	.0619	.0017	.0176	.0171	0.0000
(.02-.08-.18-.72)	16	.1037	.1120	.0236	.0405	.0525	.0077	.0081	.0280	0.0000
(.02-.08-.18-.72)	20	.0920	.0953	.0276	.0476	.0672	.0059	.0075	.0194	.0000
(.02-.08-.18-.72)	24	.0915	.0958	.0320	.0344	.0587	.0101	.0081	.0166	.0003
(.02-.08-.18-.72)	28	.0817	.0849	.0395	.0372	.0495	.0100	.0084	.0149	.0003
(.02-.08-.18-.72)	32	.0923	.0859	.0415	.0374	.0543	.0146	.0071	.0145	.0007
(.02-.08-.18-.72)	36	.0982	.0771	.0468	.0352	.0441	.0171	.0073	.0147	.0016
(.02-.08-.18-.72)	40	.1003	.0820	.0452	.0405	.0457	.0173	.0067	.0136	.0021
(.02-.08-.18-.72)	48	.1070	.0808	.0483	.0393	.0450	.0221	.0066	.0141	.0021
(.02-.08-.18-.72)	56	.1225	.0752	.0517	.0418	.0459	.0240	.0063	.0132	.0028
(.03-.07-.27-.63)	4	.1537	.1537	0.0000	.1537	.1537	0.0000	0.0000	0.0000	0.0000
(.03-.07-.27-.63)	8	.1228	.1055	.0034	.0370	.0337	0.0000	.0034	.0211	0.0000
(.03-.07-.27-.63)	12	.1137	.0928	.0148	.0539	.0397	.0011	.0106	.0094	0.0000
(.03-.07-.27-.63)	16	.1076	.0924	.0192	.0389	.0370	.0046	.0052	.0125	0.0000
(.03-.07-.27-.63)	20	.1082	.0861	.0184	.0406	.0443	.0034	.0056	.0098	0.0000
(.03-.07-.27-.63)	24	.1093	.0830	.0192	.0444	.0416	.0049	.0066	.0074	.0002
(.03-.07-.27-.63)	28	.1124	.0809	.0241	.0506	.0426	.0056	.0079	.0074	.0002
(.03-.07-.27-.63)	32	.1183	.0840	.0257	.0511	.0421	.0076	.0075	.0075	.0003
(.03-.07-.27-.63)	36	.1305	.0817	.0316	.0559	.0374	.0092	.0079	.0082	.0005
(.03-.07-.27-.63)	40	.1375	.0825	.0352	.0600	.0384	.0094	.0091	.0086	.0006
(.03-.07-.27-.63)	48	.1395	.0883	.0371	.0634	.0402	.0142	.0091	.0088	.0008
(.03-.07-.27-.63)	56	.1434	.0894	.0432	.0665	.0402	.0160	.0101	.0088	.0012
(.04-.06-.36-.54)	4	.1461	.1461	0.0000	.1461	.1461	0.0000	0.0000	0.0000	0.0000
(.04-.06-.36-.54)	8	.1076	.0803	.0029	.0310	.0267	0.0000	.0029	.0112	0.0000
(.04-.06-.36-.54)	12	.1094	.0747	.0099	.0498	.0264	.0006	.0068	.0046	0.0000
(.04-.06-.36-.54)	16	.1143	.0795	.0121	.0428	.0223	.0019	.0043	.0049	0.0000
(.04-.06-.36-.54)	20	.1251	.0836	.0112	.0575	.0305	.0015	.0040	.0040	0.0000
(.04-.06-.36-.54)	24	.1313	.0805	.0109	.0562	.0306	.0018	.0034	.0038	.0001
(.04-.06-.36-.54)	28	.1403	.0852	.0143	.0658	.0352	.0026	.0078	.0035	.0001
(.04-.06-.36-.54)	32	.1407	.0929	.0153	.0660	.0352	.0032	.0087	.0041	.0001
(.04-.06-.36-.54)	36	.1472	.0891	.0199	.0728	.0358	.0041	.0090	.0042	.0001
(.04-.06-.36-.54)	40	.1490	.0917	.0231	.0724	.0378	.0042	.0109	.0052	.0001
(.04-.06-.36-.54)	48	.1454	.0981	.0284	.0758	.0409	.0073	.0125	.0055	.0002
(.04-.06-.36-.54)	56	.1364	.1017	.0365	.0748	.0435	.0090	.0146	.0063	.0003
(.05-.05-.45-.45)	4	.1429	.1429	0.0000	.1429	.1429	0.0000	0.0000	0.0000	0.0000
(.05-.05-.45-.45)	8	.1018	.0702	.0026	.0291	.0244	0.0000	.0026	.0079	0.0000
(.05-.05-.45-.45)	12	.1084	.0674	.0080	.0494	.0222	.0005	.0035	.0030	0.0000
(.05-.05-.45-.45)	16	.1192	.0760	.0089	.0446	.0175	.0010	.0041	.0027	0.0000
(.05-.05-.45-.45)	20	.1341	.0854	.0085	.0607	.0258	.0009	.0054	.0021	0.0000
(.05-.05-.45-.45)	24	.1436	.0810	.0079	.0605	.0278	.0009	.0051	.0026	0.0000
(.05-.05-.45-.45)	28	.1521	.0894	.0111	.0705	.0360	.0016	.0077	.0023	0.0000
(.05-.05-.45-.45)	32	.1522	.0874	.0128	.0733	.0352	.0016	.0077	.0023	0.0000

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1.05-.05-.45-.45)	36	.1518	.0917	.0155	.0764	.0356	.0026	.0094	.0028	.0000	.9775
1.05-.05-.45-.45)	40	.1499	.0964	.0178	.0759	.0386	.0024	.0116	.0036	.0000	.9852
1.05-.05-.45-.45)	48	.1432	.1020	.0254	.0793	.0424	.0047	.0143	.0047	.0001	.9936
1.05-.05-.45-.45)	56	.1312	.1038	.0336	.0765	.0456	.0064	.0166	.0052	.0001	.9973
1.04-.16-.16-.61)	4	.1593	.1593	0.0000	.1593	.1593	0.0000	0.0000	0.0000	0.0000	.3467
1.04-.16-.16-.61)	8	.1371	.1204	.0052	.0490	.0207	0.0000	.0052	.0255	0.0000	.6926
1.04-.16-.16-.61)	12	.1262	.1031	.0260	.0642	.0508	.0021	.0162	.0144	0.0000	.8673
1.04-.16-.16-.61)	16	.1164	.0990	.0386	.0450	.0464	.0108	.0083	.0187	0.0000	.9445
1.04-.16-.16-.61)	20	.1031	.0910	.0346	.0348	.0498	.0097	.0083	.0158	.0002	.9771
1.04-.16-.16-.61)	24	.1084	.0822	.0415	.0479	.0459	.0154	.0094	.0106	0.010	.9906
1.04-.16-.16-.61)	28	.1077	.0807	.0503	.0518	.0432	.0176	.0076	.0114	.0013	.9961
1.04-.16-.16-.61)	32	.1370	.0847	.0492	.0593	.0449	.0233	.0082	.0099	.0018	.9984
1.04-.16-.16-.61)	36	.1425	.0861	.0560	.0593	.0396	.0243	.0088	.0109	.0026	.9994
1.04-.16-.16-.61)	40	.1410	.0923	.0540	.0637	.0417	.0260	.0085	.0109	.0032	.9997
1.04-.16-.16-.61)	48	.1398	.0966	.0542	.0689	.0425	.0274	.0096	.0099	.0044	1.0000
1.04-.16-.16-.61)	56	.1350	.0932	.0586	.0720	.0431	.0267	.0104	.0093	.0046	1.0000
1.06-.14-.24-.56)	4	.1504	.1504	0.0000	.1504	.1504	0.0000	0.0000	0.0000	0.0000	.4427
1.06-.14-.24-.56)	8	.1329	.1032	.0054	.0518	.0411	0.0000	.0054	.0160	0.0000	.7842
1.06-.14-.24-.56)	12	.1391	.1022	.0266	.0687	.0434	.0023	.0143	.0101	0.0000	.9184
1.06-.14-.24-.56)	16	.1362	.0978	.0392	.0569	.0401	.0084	.0092	.0113	0.0000	.9686
1.06-.14-.24-.56)	20	.1269	.0971	.0354	.0684	.0429	.0095	.0090	.0107	.0003	.9877
1.06-.14-.24-.56)	24	.1301	.0924	.0400	.0662	.0442	.0131	.0111	.0081	.0009	.9951
1.06-.14-.24-.56)	28	.1320	.0922	.0477	.0692	.0457	.0169	.0102	.0082	.0013	.9980
1.06-.14-.24-.56)	32	.1375	.0980	.0484	.0725	.0452	.0206	.0117	.0081	.0017	.9992
1.06-.14-.24-.56)	36	.1364	.0994	.0545	.0714	.0450	.0224	.0129	.0091	.0019	.9997
1.06-.14-.24-.56)	40	.1326	.1004	.0585	.0725	.0446	.0237	.0135	.0088	.0024	.9999
1.06-.14-.24-.56)	48	.1232	.1026	.0626	.0681	.0477	.0277	.0142	.0085	.0035	1.0000
1.06-.14-.24-.56)	56	.1148	.1025	.0681	.0635	.0486	.0282	.0147	.0088	.0041	1.0000
1.08-.12-.32-.48)	4	.1448	.1448	0.0000	.1448	.1448	0.0000	0.0000	0.0000	0.0000	.4974
1.08-.12-.32-.48)	8	.1303	.0984	.0050	.0539	.0402	0.0000	.0050	.0100	0.0000	.8177
1.08-.12-.32-.48)	12	.1505	.1034	.0248	.0763	.0406	.0026	.0128	.0063	0.0000	.9292
1.08-.12-.32-.48)	16	.1495	.1028	.0326	.0703	.0385	.0055	.0109	.0069	0.0000	.9716
1.08-.12-.32-.48)	20	.1411	.1046	.0344	.0795	.0425	.0079	.0104	.0072	.0002	.9884
1.08-.12-.32-.48)	24	.1386	.1061	.0380	.0754	.0453	.0099	.0134	.0071	.0005	.9953
1.08-.12-.32-.48)	28	.1368	.1048	.0485	.0746	.0433	.0139	.0141	.0067	.0007	.9981
1.08-.12-.32-.48)	32	.1302	.1059	.0512	.0725	.0403	.0166	.0152	.0072	.0009	.9992
1.08-.12-.32-.48)	36	.1232	.1017	.0596	.0695	.0498	.0199	.0153	.0082	.0011	.9997
1.08-.12-.32-.48)	40	.1199	.1051	.0651	.0667	.0481	.0205	.0157	.0082	.0014	.9999
1.08-.12-.32-.48)	48	.1139	.1053	.0721	.0612	.0500	.0275	.0148	.0085	.0021	1.0000
1.08-.12-.32-.48)	56	.1110	.1033	.0800	.0590	.0508	.0302	.0139	.0089	.0027	1.0000
1.10-.10-.40-.40)	4	.1429	.1429	0.0000	.1429	.1429	0.0000	0.0000	0.0000	0.0000	.5152
1.10-.10-.40-.40)	8	.1295	.0924	.0047	.0546	.0399	.0000	.0047	.0079	0.0000	.8257
1.10-.10-.40-.40)	12	.1550	.1040	.0238	.0799	.0402	.0027	.0123	.0048	0.0000	.9308
1.10-.10-.40-.40)	16	.1546	.1056	.0290	.0760	.0385	.0043	.0119	.0055	0.0000	.9718
1.10-.10-.40-.40)	20	.1450	.1088	.0339	.0832	.0432	.0071	.0110	.0062	.0002	.9885
1.10-.10-.40-.40)	24	.1400	.1110	.0373	.0773	.0447	.0087	.0144	.0066	.0003	.9933
1.10-.10-.40-.40)	28	.1359	.1109	.0495	.0742	.0534	.0123	.0158	.0063	.0004	.9981
1.10-.10-.40-.40)	32	.1298	.1075	.0541	.0712	.0499	.0149	.0165	.0071	.0006	.9992
1.10-.10-.40-.40)	36	.1177	.1008	.0633	.0680	.0503	.0194	.0156	.0079	.0008	.9997
1.10-.10-.40-.40)	40	.1163	.1069	.0688	.0637	.0494	.0196	.0157	.0079	.0009	.9999
1.10-.10-.40-.40)	48	.1115	.1061	.0766	.0594	.0509	.0279	.0144	.0088	.0014	1.0000
1.10-.10-.40-.40)	56	.1103	.1035	.0843	.0589	.0506	.0318	.0131	.0089	.0021	1.0000
1.09-.21-.21-.49)	4	.1463	.1463	0.0000	.1463	.1463	0.0000	0.0000	0.0000	0.0000	.5652
1.09-.21-.21-.49)	8	.1400	.0997	.0062	.0654	.0472	0.0000	.0062	.0118	0.0000	.8879
1.09-.21-.21-.49)	12	.1591	.1165	.0364	.0803	.0474	.0044	.0162	.0087	0.0000	.9725
1.09-.21-.21-.49)	16	.1511	.1058	.0492	.0697	.0468	.0099	.0124	.0094	0.0000	.9934
1.09-.21-.21-.49)	20	.1308	.1086	.0485	.0753	.0446	.0153	.0116	.0101	.0005	.9984
1.09-.21-.21-.49)	24	.1266	.1020	.0554	.0724	.0497	.0190	.0145	.0083	.0014	.9996
1.09-.21-.21-.49)	28	.1240	.0983	.0632	.0710	.0490	.0240	.0127	.0081	.0021	.9999
1.09-.21-.21-.49)	32	.1214	.1031	.0642	.0683	.0474	.0273	.0141	.0083	.0026	1.0000
1.09-.21-.21-.49)	36	.1174	.1038	.0724	.0658	.0515	.0291	.0153	.0093	.0030	1.0000
1.09-.21-.21-.49)	40	.1163	.1017	.0765	.0631	.0499	.0312	.0145	.0090	.0036	1.0000
1.09-.21-.21-.49)	48	.1117	.1023	.0831	.0585	.0520	.0361	.0136	.0090	.0044	1.0000
1.09-.21-.21-.49)	56	.1065	.1024	.0861	.0556	.0522	.0376	.0127	.0098	.0050	1.0000

APPENDIX G

MONTE CARLO 2×2 DATA (OVERLAP)

ESTIMATED EXACT ALPHAS FOR CTA STATISTICS

P VECTOR	N	NOMINAL ALPHA = .10		NOMINAL ALPHA = .05		NOMINAL ALPHA = .01	
		KULLBACK	PEARSON	KULLBACK	PEARSON	KULLBACK	PEARSON
(.01-.09-.09-.81)	20	.0875	.1050	.0570	.0910	.0085	.0280
(.01-.09-.09-.81)	24	.0745	.1125	.0450	.0800	.0040	.0325
(.01-.09-.09-.81)	28	.0730	.0965	.0445	.0690	.0115	.0350
(.01-.09-.09-.81)	32	.0590	.1010	.0330	.0740	.0050	.0270
(.01-.09-.09-.81)	36	.0615	.0880	.0370	.0650	.0045	.0290
(.01-.09-.09-.81)	40	.0650	.0960	.0455	.0590	.0070	.0325
(.01-.09-.09-.81)	44	.0405	.0815	.0305	.0625	.0085	.0275
(.01-.09-.09-.81)	48	.0645	.0765	.0245	.0505	.0045	.0210
(.01-.09-.09-.81)	56	.0925	.0930	.0300	.0665	.0105	.0225
(.02-.08-.18-.72)	24	.0935	.0980	.0335	.0565	.0080	.0175
(.02-.08-.18-.72)	28	.0815	.0890	.0360	.0530	.0095	.0180
(.02-.08-.18-.72)	32	.0850	.0740	.0335	.0490	.0065	.0145
(.02-.08-.18-.72)	36	.0945	.0835	.0390	.0480	.0075	.0165
(.02-.08-.18-.72)	40	.0965	.0840	.0435	.0485	.0060	.0160
(.02-.08-.18-.72)	48	.1095	.0815	.0430	.0460	.0070	.0145
(.02-.08-.18-.72)	56	.1325	.0795	.0530	.0460	.0060	.0120
(.03-.07-.27-.63)	20	.1065	.0855	.0485	.0380	.0090	.0110
(.03-.07-.27-.63)	24	.1100	.0850	.0485	.0385	.0035	.0030
(.03-.07-.27-.63)	28	.1165	.0805	.0500	.0400	.0065	.0055
(.03-.07-.27-.63)	32	.1085	.0735	.0460	.0395	.0075	.0045
(.03-.07-.27-.63)	36	.1375	.0935	.0610	.0435	.0120	.0125
(.03-.07-.27-.63)	40	.1325	.0835	.0565	.0355	.0045	.0050
(.03-.07-.27-.63)	48	.1305	.0830	.0635	.0370	.0075	.0045
(.03-.07-.27-.63)	56	.1475	.0990	.0610	.0415	.0130	.0130
(.04-.06-.36-.54)	20	.1160	.0715	.0400	.0265	.0060	.0025
(.04-.06-.36-.54)	24	.1205	.0755	.0540	.0335	.0050	.0040
(.04-.06-.36-.54)	28	.1365	.0850	.0600	.0395	.0095	.0060
(.04-.06-.36-.54)	32	.1350	.0860	.0610	.0395	.0035	.0030
(.04-.06-.36-.54)	36	.1620	.1000	.0825	.0380	.0080	.0075
(.04-.06-.36-.54)	40	.1515	.0980	.0765	.0390	.0150	.0080
(.04-.06-.36-.54)	48	.1495	.1065	.0815	.0435	.0115	.0040
(.04-.06-.36-.54)	56	.1410	.1065	.0795	.0470	.0170	.0080
(.05-.05-.45-.45)	20	.1265	.0805	.0585	.0240	.0025	.0015
(.05-.05-.45-.45)	24	.1435	.0840	.0630	.0315	.0060	.0030
(.05-.05-.45-.45)	28	.1485	.0800	.0600	.0310	.0020	.0000
(.05-.05-.45-.45)	32	.1350	.1050	.0750	.0375	.0105	.0025
(.05-.05-.45-.45)	36	.1355	.0915	.0760	.0390	.0125	.0040
(.05-.05-.45-.45)	40	.1505	.1080	.0850	.0470	.0105	.0020
(.05-.05-.45-.45)	48	.1450	.1120	.0820	.0475	.0145	.0045
(.05-.05-.45-.45)	56	.1340	.1060	.0755	.0440	.0160	.0060
(.06-.16-.16-.64)	20	.0905	.0880	.0565	.0405	.0065	.0180
(.06-.16-.16-.64)	24	.1135	.0835	.0630	.0475	.0100	.0015
(.06-.16-.16-.64)	28	.1045	.0815	.0515	.0415	.0075	.0130
(.06-.16-.16-.64)	32	.1390	.0865	.0570	.0450	.0095	.0125
(.06-.16-.16-.64)	36	.1345	.0970	.0645	.0410	.0100	.0035
(.06-.16-.16-.64)	40	.1480	.1015	.0705	.0445	.0095	.0025
(.06-.16-.16-.64)	48	.1365	.1000	.0700	.0440	.0125	.0070
(.06-.16-.16-.64)	56	.1410	.0870	.0725	.0385	.0085	.0030
(.06-.14-.24-.56)	20	.1255	.0920	.0705	.0435	.0080	.0110
(.06-.14-.24-.56)	24	.1265	.0855	.0640	.0395	.0085	.0010
(.06-.14-.24-.56)	28	.1335	.1010	.0725	.0420	.0085	.0070
(.06-.14-.24-.56)	32	.1470	.1025	.0810	.0535	.0165	.0120
(.06-.14-.24-.56)	36	.1465	.1060	.0740	.0520	.0160	.0105
(.06-.14-.24-.56)	40	.1220	.0980	.0730	.0535	.0140	.0040
(.06-.14-.24-.56)	48	.1345	.1095	.0710	.0525	.0145	.0075
(.06-.14-.24-.56)	56	.1085	.0960	.0570	.0455	.0165	.0025

(.08-.12-.32-.48)	20	.1540	.1110	.0330	.0890	.0430	.0070	.0100	.0065	.0005
(.08-.12-.32-.48)	24	.1425	.1085	.0360	.0795	.0400	.0095	.0145	.0085	.0015
(.08-.12-.32-.48)	28	.1370	.1035	.0525	.0765	.0555	.0140	.0190	.0100	.0005
(.08-.12-.32-.48)	32	.1260	.1000	.0515	.0715	.0490	.0210	.0165	.0100	.0025
(.08-.12-.32-.48)	36	.1200	.0980	.0570	.0690	.0470	.0205	.0140	.0100	.0025
(.08-.12-.32-.48)	40	.1255	.1145	.0730	.0790	.0630	.0275	.0215	.0140	.0030
(.08-.12-.32-.48)	48	.1270	.1130	.0750	.0875	.0565	.0270	.0165	.0075	.0010
(.08-.12-.32-.48)	56	.1015	.0920	.0700	.0500	.0410	.0255	.0065	.0050	.0010
(.10-.10-.40-.60)	20	.1420	.1085	.0345	.0775	.0460	.0085	.0105	.0055	.0000
(.10-.10-.40-.60)	24	.1375	.1135	.0440	.0755	.0535	.0125	.0135	.0050	.0000
(.10-.10-.40-.60)	28	.1355	.1085	.0420	.0740	.0475	.0095	.0150	.0080	.0005
(.10-.10-.40-.60)	32	.1330	.1110	.0575	.0700	.0475	.0150	.0155	.0080	.0005
(.10-.10-.40-.60)	36	.1275	.1100	.0705	.0730	.0540	.0180	.0180	.0095	.0015
(.10-.10-.40-.60)	40	.1055	.0960	.0645	.0630	.0505	.0195	.0195	.0100	.0010
(.10-.10-.40-.60)	48	.1075	.1015	.0735	.0530	.0455	.0265	.0085	.0040	.0005
(.10-.10-.40-.60)	56	.1140	.1105	.0860	.0600	.0520	.0315	.0155	.0105	.0030
(.09-.21-.21-.49)	20	.1370	.1125	.0495	.0750	.0420	.0165	.0090	.0080	.0005
(.09-.21-.21-.49)	24	.1270	.1095	.0520	.0745	.0550	.0180	.0105	.0110	.0015
(.09-.21-.21-.49)	28	.1260	.1070	.0690	.0730	.0525	.0245	.0115	.0085	.0030
(.09-.21-.21-.49)	32	.1205	.1015	.0620	.0670	.0505	.0250	.0100	.0065	.0015
(.09-.21-.21-.49)	36	.1175	.1020	.0705	.0670	.0590	.0330	.0195	.0125	.0040
(.09-.21-.21-.49)	40	.1225	.1120	.0875	.0655	.0530	.0360	.0160	.0100	.0030
(.09-.21-.21-.49)	48	.1130	.1035	.0850	.0525	.0425	.0325	.0120	.0095	.0050
(.09-.21-.21-.49)	56	.1040	.1010	.0880	.0530	.0510	.0395	.0115	.0105	.0060
(.12-.18-.28-.42)	20	.1270	.1075	.0535	.0670	.0435	.0180	.0155	.0110	.0015
(.12-.18-.28-.42)	24	.1340	.1200	.0690	.0730	.0530	.0230	.0145	.0075	.0020
(.12-.18-.28-.42)	28	.1120	.0945	.0495	.0645	.0460	.0350	.0130	.0075	.0025
(.12-.18-.28-.42)	32	.1255	.1080	.0820	.0645	.0535	.0350	.0130	.0075	.0025
(.12-.18-.28-.42)	36	.1280	.1100	.0940	.0750	.0655	.0410	.0185	.0115	.0045
(.12-.18-.28-.42)	40	.1045	.0960	.0805	.0500	.0455	.0330	.0110	.0070	.0030
(.12-.18-.28-.42)	48	.0995	.0975	.0805	.0460	.0430	.0345	.0135	.0095	.0040
(.12-.18-.28-.42)	56	.1175	.1140	.1060	.0675	.0640	.0550	.0145	.0125	.0075
(.13-.15-.35-.35)	20	.1230	.1110	.0625	.0720	.0500	.0165	.0155	.0090	.0010
(.13-.15-.35-.35)	24	.1035	.0945	.0620	.0595	.0505	.0215	.0130	.0085	.0010
(.13-.15-.35-.35)	28	.1220	.1060	.0820	.0595	.0515	.0245	.0130	.0070	.0015
(.13-.15-.35-.35)	32	.1130	.0935	.0810	.0505	.0455	.0260	.0135	.0100	.0005
(.13-.15-.35-.35)	36	.1080	.1045	.0920	.0650	.0575	.0395	.0125	.0080	.0030
(.13-.15-.35-.35)	40	.1175	.1130	.1005	.0620	.0565	.0385	.0120	.0095	.0045
(.13-.15-.35-.35)	48	.1060	.1035	.0925	.0540	.0485	.0415	.0110	.0070	.0025
(.13-.15-.35-.35)	56	.1075	.1040	.0965	.0585	.0535	.0460	.0150	.0085	.0015
(.16-.24-.24-.36)	20	.1310	.1270	.0835	.0675	.0555	.0255	.0175	.0130	.0010
(.16-.24-.24-.36)	24	.1160	.1125	.0855	.0660	.0640	.0340	.0170	.0090	.0020
(.16-.24-.24-.36)	28	.1170	.1035	.0650	.0570	.0510	.0335	.0130	.0085	.0015
(.16-.24-.24-.36)	32	.1170	.1095	.0950	.0570	.0510	.0420	.0105	.0065	.0045
(.16-.24-.24-.36)	36	.1105	.1060	.0975	.0540	.0485	.0360	.0085	.0065	.0020
(.16-.24-.24-.36)	40	.1025	.0995	.0945	.0530	.0515	.0340	.0110	.0065	.0045
(.16-.24-.24-.36)	48	.1045	.1030	.0960	.0475	.0465	.0440	.0090	.0080	.0045
(.16-.24-.24-.36)	56	.1085	.1040	.1015	.0595	.0580	.0530	.0160	.0135	.0100
(.20-.20-.30-.30)	20	.1305	.1335	.0925	.0705	.0575	.0315	.0140	.0105	.0020
(.20-.20-.30-.30)	24	.1130	.1090	.0790	.0620	.0575	.0370	.0175	.0135	.0040
(.20-.20-.30-.30)	28	.1130	.1060	.0890	.0600	.0545	.0350	.0130	.0105	.0015
(.20-.20-.30-.30)	32	.1070	.0995	.0865	.0495	.0495	.0330	.0125	.0095	.0030
(.20-.20-.30-.30)	36	.1260	.1195	.1100	.0690	.0680	.0530	.0115	.0085	.0050
(.20-.20-.30-.30)	40	.0945	.0925	.0915	.0510	.0490	.0435	.0060	.0050	.0045
(.20-.20-.30-.30)	48	.1010	.1000	.0960	.0490	.0465	.0445	.0130	.0100	.0055
(.20-.20-.30-.30)	56	.1020	.0985	.0945	.0545	.0525	.0500	.0145	.0115	.0090
(.25-.25-.25-.25)	20	.1210	.1200	.0945	.0540	.0500	.0340	.0155	.0125	.0020
(.25-.25-.25-.25)	24	.1225	.1150	.0890	.0650	.0645	.0485	.0145	.0130	.0045
(.25-.25-.25-.25)	28	.0985	.0920	.0865	.0515	.0505	.0325	.0095	.0075	.0020
(.25-.25-.25-.25)	32	.1150	.1120	.1035	.0590	.0545	.0470	.0140	.0100	.0045
(.25-.25-.25-.25)	36	.1035	.0980	.0960	.0525	.0500	.0405	.0100	.0095	.0060
(.25-.25-.25-.25)	40	.1075	.1050	.1045	.0590	.0590	.0545	.0125	.0105	.0075
(.25-.25-.25-.25)	48	.1025	.1020	.1005	.0540	.0530	.0500	.0115	.0110	.0075
(.25-.25-.25-.25)	56	.1120	.1095	.1035	.0515	.0495	.0495	.0115	.0095	.0095

APPENDIX H

MONTE CARLO 2×3 DATA

ESTIMATED EXACT ALPHAS FOR CTA STATISTICS

P VECTOR	N	NOMINAL ALPHA = .10				NOMINAL ALPHA = .05				NOMINAL ALPHA = .01			
		KULLBACK	PEARSON	GSKI		KULLBACK	PEARSON	GSKI		KULLBACK	PEARSON	GSKI	
(.01-.01-.08-.09-.09-.72)	12	.0885	.1775	0.0000		.0695	.0680	0.0000		.0050	.0490	0.0000	
(.01-.01-.08-.09-.09-.72)	16	.1060	.1360	0.0025		.0330	.0930	0.0000		.0025	.0230	0.0000	
(.01-.01-.08-.09-.09-.72)	24	.0820	.1460	0.0120		.0320	.0890	0.0000		.0065	.0410	0.0000	
(.01-.01-.08-.09-.09-.72)	30	.0860	.1255	0.0140		.0380	.1040	0.0020		.0065	.0490	0.0000	
(.01-.01-.08-.09-.09-.72)	36	.0610	.1145	0.0250		.0260	.0635	0.0040		.0040	.0200	0.0000	
(.01-.01-.08-.09-.09-.72)	48	.0620	.0960	0.0315		.0265	.0595	0.0105		.0050	.0215	0.0010	
(.01-.01-.08-.09-.09-.72)	60	.0675	.0905	0.0475		.0290	.0605	0.0200		.0040	.0230	0.0015	
(.01-.01-.08-.09-.09-.72)	72	.0815	.0920	0.0500		.0305	.0560	0.0220		.0050	.0210	0.0020	
(.01-.01-.08-.09-.09-.72)	84	.0765	.0825	0.0525		.0345	.0525	0.0285		.0055	.0200	0.0040	
(.01-.01-.08-.09-.09-.72)	96	.0845	.0725	0.0600		.0320	.0470	0.0225		.0075	.0155	0.0050	
(.01-.01-.08-.09-.09-.72)	12	.0775	.1295	0.0000		.0470	0.0000	0.0000		.0040	.0270	0.0000	
(.01-.03-.06-.09-.27-.54)	16	.1015	.1105	0.0020		.0425	.0670	0.0000		.0060	.0195	0.0000	
(.01-.03-.06-.09-.27-.54)	24	.0890	.1040	0.0030		.0355	.0615	0.0010		.0055	.0195	0.0000	
(.01-.03-.06-.09-.27-.54)	30	.0955	.0965	0.0115		.0475	.0665	0.0010		.0080	.0260	0.0000	
(.01-.03-.06-.09-.27-.54)	36	.1095	.0995	0.0160		.0435	.0535	0.0045		.0065	.0195	0.0000	
(.01-.03-.06-.09-.27-.54)	48	.0975	.0830	0.0190		.0430	.0475	0.0035		.0110	.0135	0.0000	
(.01-.03-.06-.09-.27-.54)	60	.1145	.0950	0.0345		.0505	.0505	0.0115		.0075	.0175	0.0005	
(.01-.03-.06-.09-.27-.54)	72	.1215	.0995	0.0455		.0615	.0370	0.0205		.0090	.0170	0.0010	
(.01-.03-.06-.09-.27-.54)	84	.1090	.0895	0.0430		.0515	.0535	0.0145		.0090	.0135	0.0010	
(.01-.03-.06-.09-.27-.54)	96	.1070	.0875	0.0410		.0425	.0425	0.0095		.0095	.0135	0.0010	
(.02-.02-.06-.18-.18-.54)	12	.0940	.1525	0.0000		.0580	.0585	0.0000		.0060	.0310	0.0000	
(.02-.02-.06-.18-.18-.54)	16	.0880	.0995	0.0015		.0315	.0545	0.0000		.0065	.0135	0.0000	
(.02-.02-.06-.18-.18-.54)	24	.0845	.0840	0.0135		.0425	.0535	0.0030		.0075	.0140	0.0000	
(.02-.02-.06-.18-.18-.54)	30	.0970	.0935	0.0155		.0405	.0470	0.0050		.0070	.0115	0.0000	
(.02-.02-.06-.18-.18-.54)	36	.1105	.0900	0.0220		.0475	.0430	0.0080		.0065	.0095	0.0000	
(.02-.02-.06-.18-.18-.54)	48	.1185	.0890	0.0250		.0545	.0460	0.0085		.0070	.0140	0.0005	
(.02-.02-.06-.18-.18-.54)	60	.1290	.0775	0.0355		.0630	.0460	0.0160		.0105	.0130	0.0020	
(.02-.02-.06-.18-.18-.54)	72	.1380	.0850	0.0380		.0640	.0435	0.0140		.0075	.0095	0.0020	
(.02-.02-.06-.18-.18-.54)	84	.1305	.0830	0.0400		.0375	.0585	0.0055		.0045	.0055	0.0005	
(.02-.02-.06-.18-.18-.54)	96	.0915	.1210	0.0000		.0410	.0415	0.0000		.0030	.0160	0.0000	
(.02-.04-.04-.18-.36-.36)	12	.0920	.0765	0.0005		.0345	.0380	0.0005		.0050	.0080	0.0000	
(.02-.04-.04-.18-.36-.36)	16	.0895	.0750	0.0040		.0280	.0280	0.0010		.0025	.0065	0.0000	
(.02-.04-.04-.18-.36-.36)	24	.1085	.0740	0.0075		.0420	.0335	0.0000		.0025	.0055	0.0000	
(.02-.04-.04-.18-.36-.36)	36	.1265	.0870	0.0125		.0550	.0395	0.0030		.0075	.0085	0.0000	
(.02-.04-.04-.18-.36-.36)	48	.1395	.0875	0.0185		.0605	.0405	0.0045		.0080	.0090	0.0000	
(.02-.04-.04-.18-.36-.36)	60	.1465	.0950	0.0185		.0730	.0510	0.0080		.0120	.0090	0.0005	
(.02-.04-.04-.18-.36-.36)	72	.1460	.0965	0.0255		.0725	.0475	0.0080		.0110	.0060	0.0005	
(.02-.04-.04-.18-.36-.36)	84	.1245	.0955	0.0310		.0655	.0465	0.0110		.0095	.0085	0.0005	
(.02-.04-.04-.18-.36-.36)	96	.1315	.0910	0.0370		.0635	.0455	0.0125		.0165	.0105	0.0000	
(.02-.04-.14-.08-.16-.56)	12	.0980	.1235	0.0000		.0550	.0525	0.0000		.0100	.0210	0.0000	
(.02-.04-.14-.08-.16-.56)	16	.1115	.0955	0.0050		.0405	.0500	0.0000		.0055	.0120	0.0000	
(.02-.04-.14-.08-.16-.56)	24	.1135	.1005	0.0170		.0585	.0505	0.0025		.0120	.0190	0.0000	
(.02-.04-.14-.08-.16-.56)	30	.1185	.1005	0.0220		.0545	.0550	0.0060		.0110	.0145	0.0000	
(.02-.04-.14-.08-.16-.56)	36	.1235	.1045	0.0355		.0635	.0575	0.0130		.0075	.0120	0.0005	
(.02-.04-.14-.08-.16-.56)	48	.1195	.0895	0.0435		.0560	.0560	0.0145		.0050	.0105	0.0000	
(.02-.04-.14-.08-.16-.56)	60	.1265	.0895	0.0485		.0610	.0500	0.0210		.0120	.0130	0.0015	
(.02-.04-.14-.08-.16-.56)	72	.1330	.0940	0.0535		.0620	.0455	0.0220		.0100	.0100	0.0015	
(.02-.04-.14-.08-.16-.56)	84	.1345	.0980	0.0535		.0605	.0470	0.0200		.0095	.0095	0.0020	
(.02-.04-.14-.08-.16-.56)	96	.1280	.0925	0.0505		.0605	.0410	0.0115		.0085	.0085	0.0040	
(.02-.08-.10-.08-.32-.40)	12	.1175	.1080	0.0000		.0485	.0480	0.0000		.0075	.0110	0.0000	
(.02-.08-.10-.08-.32-.40)	16	.1275	.0975	0.0035		.0605	.0455	0.0000		.0075	.0110	0.0000	
(.02-.08-.10-.08-.32-.40)	24	.1285	.0975	0.0060		.0615	.0525	0.0015		.0125	.0110	0.0000	
(.02-.08-.10-.08-.32-.40)	30	.1240	.0960	0.0115		.0620	.0490	0.0040		.0070	.0105	0.0005	
(.02-.08-.10-.08-.32-.40)	36	.1115	.0945	0.0185		.0575	.0460	0.0040		.0130	.0105	0.0000	
(.02-.08-.10-.08-.32-.40)	48	.1080	.0900	0.0315		.0525	.0430	0.0105		.0120	.0105	0.0005	

60	(.02, .08, .10, .08, .32, .40)	.1220	.1040	.0520	.0620	.0525	.0170	.0130	.0110	.0020
72	(.02, .08, .10, .08, .32, .40)	.1190	.0945	.0600	.0585	.0530	.0245	.0120	.0100	.0015
84	(.02, .08, .10, .08, .32, .40)	.1195	.0915	.0625	.0540	.0480	.0260	.0100	.0095	.0010
96	(.02, .08, .10, .08, .32, .40)	.1205	.0960	.0725	.0590	.0495	.0265	.0075	.0095	.0020
12	(.06, .06, .10, .16, .24, .40)	.1200	.1100	0.0000	.0560	.0450	0.0000	.0095	.0060	0.0000
18	(.06, .06, .10, .16, .24, .40)	.1325	.0930	.0045	.0630	.0390	.0010	.0030	.0065	0.0000
24	(.06, .06, .10, .16, .24, .40)	.1290	.0900	.0115	.0620	.0430	.0030	.0095	.0085	0.0000
30	(.06, .06, .10, .16, .24, .40)	.1200	.0795	.0255	.0600	.0435	.0095	.0105	.0080	0.0000
36	(.06, .06, .10, .16, .24, .40)	.1570	.1010	.0305	.0695	.0470	.0085	.0170	.0110	.0005
48	(.06, .06, .10, .16, .24, .40)	.1265	.0910	.0310	.0640	.0395	.0115	.0140	.0070	0.0000
60	(.06, .06, .10, .16, .24, .40)	.1195	.0905	.0475	.0650	.0460	.0165	.0115	.0070	.0020
72	(.06, .06, .10, .16, .24, .40)	.1275	.0995	.0580	.0675	.0465	.0100	.0140	.0065	.0015
84	(.06, .06, .10, .16, .24, .40)	.1230	.1015	.0655	.0650	.0485	.0240	.0145	.0095	.0045
96	(.06, .06, .10, .16, .24, .40)	.1160	.0960	.0620	.0660	.0510	.0290	.0125	.0095	.0030
12	(.06, .06, .08, .24, .24, .32)	.1285	.1080	0.0000	.0550	.0415	0.0000	.0075	.0080	0.0000
18	(.06, .06, .08, .24, .24, .32)	.1395	.0865	.0065	.0610	.0415	0.0000	.0075	.0080	0.0000
24	(.06, .06, .08, .24, .24, .32)	.1365	.0910	.0145	.0635	.0385	.0035	.0100	.0055	0.0000
30	(.06, .06, .08, .24, .24, .32)	.1525	.0945	.0195	.0715	.0460	.0035	.0105	.0065	0.0000
36	(.06, .06, .08, .24, .24, .32)	.1485	.1030	.0350	.0775	.0505	.0085	.0140	.0080	.0005
48	(.06, .06, .08, .24, .24, .32)	.1335	.0945	.0360	.0705	.0450	.0125	.0195	.0080	.0020
60	(.06, .06, .08, .24, .24, .32)	.1190	.0955	.0510	.0655	.0475	.0170	.0095	.0095	.0020
72	(.06, .06, .08, .24, .24, .32)	.1305	.1100	.0675	.0675	.0560	.0215	.0170	.0095	.0020
84	(.06, .06, .08, .24, .24, .32)	.1095	.0955	.0635	.0615	.0490	.0255	.0155	.0080	.0015
96	(.06, .06, .08, .24, .24, .32)	.1075	.0960	.0765	.0545	.0490	.0285	.0130	.0090	.0045
12	(.03, .03, .24, .07, .07, .56)	.1035	.1020	0.0000	.0475	.0340	0.0000	.0075	.0100	0.0000
18	(.03, .03, .24, .07, .07, .56)	.1200	.0850	.0025	.0400	.0310	0.0000	.0050	.0040	0.0000
24	(.03, .03, .24, .07, .07, .56)	.1235	.0850	.0075	.0530	.0380	.0020	.0100	.0080	0.0000
30	(.03, .03, .24, .07, .07, .56)	.1210	.0890	.0155	.0595	.0435	.0035	.0075	.0070	.0005
36	(.03, .03, .24, .07, .07, .56)	.1285	.0830	.0155	.0605	.0415	.0060	.0075	.0070	.0005
48	(.03, .03, .24, .07, .07, .56)	.1465	.0905	.0310	.0685	.0460	.0080	.0085	.0095	0.0000
60	(.03, .03, .24, .07, .07, .56)	.1420	.0880	.0280	.0700	.0420	.0080	.0125	.0090	.0010
72	(.03, .03, .24, .07, .07, .56)	.1505	.0980	.0490	.0690	.0520	.0155	.0150	.0120	.0015
84	(.03, .03, .24, .07, .07, .56)	.1315	.0895	.0400	.0735	.0485	.0185	.0110	.0060	.0015
96	(.03, .03, .24, .07, .07, .56)	.1430	.1010	.0530	.0755	.0475	.0195	.0180	.0060	.0025
12	(.03, .09, .18, .07, .21, .42)	.1490	.1120	0.0000	.0680	.0505	0.0000	.0105	.0095	0.0000
18	(.03, .09, .18, .07, .21, .42)	.1435	.1030	.0075	.0745	.0475	.0010	.0160	.0110	0.0000
24	(.03, .09, .18, .07, .21, .42)	.1450	.0995	.0155	.0660	.0415	.0020	.0130	.0060	0.0000
30	(.03, .09, .18, .07, .21, .42)	.1375	.1015	.0345	.0710	.0560	.0090	.0150	.0100	0.0000
36	(.03, .09, .18, .07, .21, .42)	.1235	.0960	.0300	.0685	.0415	.0140	.0175	.0105	.0015
48	(.03, .09, .18, .07, .21, .42)	.1275	.0965	.0470	.0590	.0480	.0240	.0130	.0100	.0010
60	(.03, .09, .18, .07, .21, .42)	.1140	.1085	.0515	.0600	.0445	.0155	.0100	.0060	.0005
72	(.03, .09, .18, .07, .21, .42)	.1400	.1075	.0670	.0735	.0490	.0240	.0100	.0085	.0030
84	(.03, .09, .18, .07, .21, .42)	.1190	.0975	.0640	.0550	.0425	.0205	.0090	.0065	.0025
96	(.03, .09, .18, .07, .21, .42)	.1305	.1075	.0705	.0705	.0560	.0325	.0150	.0115	.0040
12	(.06, .06, .18, .14, .14, .42)	.1135	.0855	0.0000	.0480	.0305	0.0000	.0080	.0065	0.0000
18	(.06, .06, .18, .14, .14, .42)	.1480	.0850	.0135	.0720	.0515	.0030	.0165	.0080	0.0000
24	(.06, .06, .18, .14, .14, .42)	.1390	.0875	.0155	.0725	.0470	.0050	.0160	.0095	0.0000
30	(.06, .06, .18, .14, .14, .42)	.1510	.1015	.0305	.0720	.0475	.0080	.0105	.0065	.0005
36	(.06, .06, .18, .14, .14, .42)	.1440	.1005	.0350	.0755	.0515	.0150	.0120	.0080	0.0000
48	(.06, .06, .18, .14, .14, .42)	.1310	.0995	.0495	.0715	.0460	.0195	.0125	.0090	.0005
60	(.06, .06, .18, .14, .14, .42)	.1130	.0945	.0500	.0550	.0390	.0200	.0155	.0080	.0020
72	(.06, .06, .18, .14, .14, .42)	.1115	.0990	.0625	.0630	.0490	.0275	.0175	.0090	.0055
84	(.06, .06, .18, .14, .14, .42)	.1185	.1020	.0735	.0575	.0500	.0305	.0150	.0080	.0030
96	(.06, .06, .18, .14, .14, .42)	.1105	.1045	.0740	.0610	.0485	.0315	.0145	.0105	.0045
12	(.06, .12, .12, .14, .28, .28)	.1540	.1040	0.0000	.0675	.0515	0.0000	.0140	.0065	0.0000
18	(.06, .12, .12, .14, .28, .28)	.1575	.1005	.0105	.0830	.0460	.0010	.0180	.0080	0.0000
24	(.06, .12, .12, .14, .28, .28)	.1465	.0970	.0250	.0765	.0500	.0035	.0145	.0080	0.0000
30	(.06, .12, .12, .14, .28, .28)	.1390	.1030	.0415	.0795	.0520	.0105	.0165	.0090	.0005
36	(.06, .12, .12, .14, .28, .28)	.1340	.1070	.0475	.0730	.0550	.0175	.0170	.0115	.0010
48	(.06, .12, .12, .14, .28, .28)	.1325	.1045	.0635	.0730	.0495	.0240	.0175	.0095	.0025
60	(.06, .12, .12, .14, .28, .28)	.1045	.0925	.0595	.0550	.0405	.0240	.0100	.0055	.0025
72	(.06, .12, .12, .14, .28, .28)	.1200	.1110	.0830	.0625	.0550	.0430	.0185	.0125	.0035
84	(.06, .12, .12, .14, .28, .28)	.1095	.1005	.0820	.0590	.0515	.0370	.0130	.0070	.0035
96	(.06, .12, .12, .14, .28, .28)	.1055	.1020	.0880	.0565	.0455	.0415	.0125	.0105	.0035

(.04..08..28..06..12..42)	12	.1310	.0815	0.0000	.0270	.0365	0.0000	.0090	.0045	0.0000
(.04..08..28..06..12..42)	16	.1390	.0680	.0035	.0655	.0325	.0015	.0070	.0030	0.0000
(.04..08..28..06..12..42)	24	.1455	.0815	.0110	.0680	.0365	.0025	.0145	.0050	0.0000
(.04..08..28..06..12..42)	30	.1420	.0825	.0170	.0665	.0340	.0045	.0130	.0045	0.0000
(.04..08..28..06..12..42)	36	.1290	.0955	.0235	.0745	.0360	.0055	.0125	.0075	.0005
(.04..08..28..06..12..42)	48	.1250	.0885	.0345	.0610	.0455	.0110	.0155	.0060	.0025
(.04..08..28..06..12..42)	60	.1295	.0995	.0430	.0690	.0440	.0150	.0175	.0080	.0020
(.04..08..28..06..12..42)	72	.1090	.0875	.0500	.0605	.0475	.0195	.0095	.0095	.0020
(.04..08..28..06..12..42)	84	.1240	.1045	.0595	.0680	.0485	.0230	.0145	.0095	.0035
(.04..08..28..06..12..42)	96	.1215	.1050	.0745	.0620	.0500	.0275	.0125	.0085	.0020
(.04..16..20..06..24..30)	12	.1505	.0910	0.0000	.0730	.0385	0.0000	.0120	.0035	0.0000
(.04..16..20..06..24..30)	16	.1450	.0905	.0135	.0720	.0400	.0005	.0165	.0045	0.0000
(.04..16..20..06..24..30)	24	.1320	.0950	.0235	.0655	.0490	.0070	.0145	.0065	0.0000
(.04..16..20..06..24..30)	30	.1240	.0920	.0365	.0655	.0435	.0115	.0135	.0075	.0005
(.04..16..20..06..24..30)	36	.1335	.1020	.0435	.0620	.0410	.0130	.0115	.0050	.0005
(.04..16..20..06..24..30)	48	.1470	.1125	.0610	.0700	.0480	.0245	.0150	.0115	.0020
(.04..16..20..06..24..30)	60	.1240	.1030	.0600	.0685	.0400	.0270	.0115	.0060	.0015
(.04..16..20..06..24..30)	72	.1150	.0960	.0600	.0565	.0400	.0230	.0110	.0060	.0035
(.04..16..20..06..24..30)	84	.1050	.0935	.0610	.0560	.0430	.0235	.0090	.0065	.0020
(.04..16..20..06..24..30)	96	.1295	.1180	.0875	.0680	.0380	.0395	.0205	.0155	.0090
(.08..12..20..12..18..30)	12	.1550	.0925	.0005	.0745	.0435	0.0000	.0165	.0035	0.0000
(.08..12..20..12..18..30)	16	.1500	.0885	.0145	.0745	.0430	.0015	.0155	.0045	0.0000
(.08..12..20..12..18..30)	24	.1455	.0990	.0325	.0785	.0490	.0080	.0125	.0035	0.0000
(.08..12..20..12..18..30)	30	.1300	.1005	.0395	.0710	.0505	.0125	.0190	.0075	.0005
(.08..12..20..12..18..30)	36	.1285	.1025	.0575	.0705	.0475	.0180	.0140	.0080	.0010
(.08..12..20..12..18..30)	48	.1160	.1005	.0715	.0605	.0500	.0255	.0115	.0080	.0035
(.08..12..20..12..18..30)	60	.1060	.0895	.0635	.0460	.0385	.0240	.0120	.0065	.0020
(.08..12..20..12..18..30)	72	.1015	.1015	.0830	.0545	.0455	.0375	.0135	.0095	.0060
(.08..12..20..12..18..30)	84	.1100	.1050	.0930	.0540	.0480	.0395	.0115	.0085	.0055
(.08..12..20..12..18..30)	96	.1045	.1010	.0870	.0535	.0500	.0395	.0165	.0120	.0085
(.12..12..16..18..18..24)	12	.1680	.1030	.0005	.0835	.0415	0.0000	.0150	.0035	0.0000
(.12..12..16..18..18..24)	16	.1635	.1070	.0200	.0865	.0520	.0045	.0255	.0080	0.0000
(.12..12..16..18..18..24)	24	.1420	.1065	.0435	.0770	.0500	.0115	.0185	.0095	0.0000
(.12..12..16..18..18..24)	30	.1220	.0985	.0530	.0685	.0465	.0180	.0175	.0075	.0005
(.12..12..16..18..18..24)	36	.1225	.1100	.0710	.0640	.0530	.0280	.0140	.0085	.0015
(.12..12..16..18..18..24)	48	.1135	.1040	.0835	.0580	.0505	.0315	.0125	.0085	.0025
(.12..12..16..18..18..24)	60	.1170	.1090	.0915	.0635	.0595	.0385	.0125	.0090	.0040
(.12..12..16..18..18..24)	72	.1070	.1030	.0910	.0540	.0515	.0425	.0130	.0115	.0050
(.12..12..16..18..18..24)	84	.1020	.0985	.0880	.0485	.0455	.0390	.0090	.0085	.0045
(.12..12..16..18..18..24)	96	.0905	.0870	.0800	.0455	.0440	.0380	.0070	.0055	.0030
(.05..05..10..05..05..40)	12	.1225	.0620	0.0000	.0405	.0175	0.0000	.0040	.0020	0.0000
(.05..05..10..05..05..40)	16	.1425	.0550	.0025	.0495	.0155	.0005	.0055	.0020	0.0000
(.05..05..10..05..05..40)	24	.1540	.0730	.0045	.0670	.0305	0.0000	.0100	.0035	0.0000
(.05..05..10..05..05..40)	30	.1505	.0830	.0020	.0765	.0280	.0025	.0100	.0020	0.0000
(.05..05..10..05..05..40)	36	.1675	.0935	.0095	.0905	.0325	.0025	.0155	.0030	0.0000
(.05..05..10..05..05..40)	48	.1675	.1055	.0130	.0930	.0425	.0130	.0155	.0035	0.0000
(.05..05..10..05..05..40)	60	.1410	.1045	.0275	.0870	.0480	.0400	.0155	.0050	0.0000
(.05..05..10..05..05..40)	72	.1420	.1105	.0360	.0785	.0475	.0100	.0230	.0100	0.0000
(.05..05..10..05..05..40)	84	.1295	.0970	.0370	.0645	.0430	.0075	.0170	.0070	.0005
(.05..05..10..05..05..40)	96	.1240	.1120	.0605	.0640	.0480	.0155	.0155	.0075	.0010
(.05..15..30..05..15..30)	12	.1355	.0825	0.0000	.0630	.0345	0.0000	.0100	.0025	0.0000
(.05..15..30..05..15..30)	16	.1240	.0735	.0125	.0650	.0330	.0000	.0110	.0015	0.0000
(.05..15..30..05..15..30)	24	.1235	.0755	.0165	.0585	.0305	.0035	.0100	.0020	0.0000
(.05..15..30..05..15..30)	30	.1355	.0935	.0380	.0650	.0430	.0120	.0130	.0045	0.0000
(.05..15..30..05..15..30)	36	.1265	.0915	.0415	.0710	.0490	.0130	.0165	.0075	.0010
(.05..15..30..05..15..30)	48	.1340	.1000	.0500	.0695	.0435	.0150	.0095	.0055	.0005
(.05..15..30..05..15..30)	60	.1270	.1050	.0695	.0720	.0530	.0235	.0190	.0090	.0025
(.05..15..30..05..15..30)	72	.1285	.1130	.0665	.0695	.0495	.0240	.0130	.0065	.0020
(.05..15..30..05..15..30)	84	.1160	.0985	.0705	.0670	.0545	.0335	.0160	.0100	.0030
(.05..15..30..05..15..30)	96	.1055	.0940	.0670	.0535	.0425	.0295	.0115	.0070	.0035
(.10..10..30..10..10..30)	12	.1550	.0840	0.0000	.0715	.0315	0.0000	.0090	.0030	0.0000
(.10..10..30..10..10..30)	16	.1720	.0960	.0170	.0815	.0405	.0035	.0150	.0035	0.0000
(.10..10..30..10..10..30)	24	.1535	.1005	.0200	.0785	.0405	.0030	.0140	.0055	0.0000
(.10..10..30..10..10..30)	30	.1515	.1090	.0345	.0780	.0495	.0125	.0180	.0110	0.0000

(.10..10..30..10..10..30)	36	.1275	.0980	.0440	.0700	.0315	.0135	.0220	.0110	.0010
(.10..10..30..10..10..30)	48	.1315	.1165	.0695	.0695	.0345	.0235	.0190	.0105	.0020
(.10..10..30..10..10..30)	60	.1215	.1110	.0705	.0615	.0520	.0330	.0185	.0100	.0025
(.10..10..30..10..10..30)	72	.1000	.0925	.0725	.0500	.0430	.0240	.0110	.0080	.0010
(.10..10..30..10..10..30)	84	.1210	.1105	.0905	.0370	.0495	.0355	.0120	.0085	.0035
(.10..10..30..10..10..30)	96	.1010	.0950	.0810	.0310	.0465	.0400	.0130	.0100	.0040
(.10..20..20..10..20..20)	12	.1715	.1115	0.0000	.0890	.0540	0.0000	.0215	.0025	0.0000
(.10..20..20..10..20..20)	16	.1485	.0990	.0165	.0800	.0445	.0040	.0175	.0035	0.0000
(.10..20..20..10..20..20)	24	.1475	.1085	.0400	.0780	.0515	.0110	.0190	.0070	.0005
(.10..20..20..10..20..20)	30	.1395	.1080	.0505	.0730	.0535	.0170	.0190	.0085	.0010
(.10..20..20..10..20..20)	36	.1200	.0975	.0590	.0625	.0460	.0230	.0165	.0070	.0025
(.10..20..20..10..20..20)	48	.1150	.1070	.0765	.0625	.0515	.0280	.0135	.0100	.0045
(.10..20..20..10..20..20)	60	.1145	.1045	.0870	.0575	.0530	.0365	.0145	.0115	.0055
(.10..20..20..10..20..20)	72	.1195	.1110	.0925	.0575	.0515	.0380	.0120	.0110	.0050
(.10..20..20..10..20..20)	84	.1110	.1060	.0945	.0595	.0550	.0440	.0120	.0100	.0060
(.10..20..20..10..20..20)	96	.0985	.0970	.0895	.0480	.0465	.0380	.0105	.0090	.0065
(.17..17..17..17..17..17)	12	.1685	.0975	.0005	.0855	.0490	0.0000	.0185	.0025	0.0000
(.17..17..17..17..17..17)	16	.1500	.0970	.0190	.0810	.0465	.0015	.0225	.0085	0.0000
(.17..17..17..17..17..17)	24	.1275	.0980	.0400	.0680	.0430	.0115	.0160	.0075	0.0000
(.17..17..17..17..17..17)	30	.1190	.1040	.0495	.0640	.0410	.0120	.0130	.0065	0.0000
(.17..17..17..17..17..17)	36	.1195	.1065	.0735	.0640	.0515	.0255	.0145	.0095	.0010
(.17..17..17..17..17..17)	48	.1230	.1140	.0975	.0640	.0425	.0425	.0145	.0110	.0030
(.17..17..17..17..17..17)	60	.1080	.1030	.0930	.0590	.0550	.0425	.0190	.0150	.0080
(.17..17..17..17..17..17)	72	.1140	.1115	.1015	.0610	.0595	.0465	.0125	.0105	.0065
(.17..17..17..17..17..17)	84	.1040	.1005	.0895	.0540	.0495	.0425	.0130	.0095	.0040
(.17..17..17..17..17..17)	96	.0995	.0955	.0890	.0470	.0425	.0385	.0115	.0105	.0080

APPENDIX I

MINIMUM AND MAXIMUM SIGNIFICANCE LEVELS
FOR MINIMUM CELL EXPECTATION INTERVALS (MCEI)

MINIMUM AND MAXIMUM SIGNIFICANCE LEVELS
FOR MINIMUM CELL EXPECTATION INTERVALS(MCEI)

TABLE: 2X2

NOMINAL ALPHA = .10			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0595,.1505)	(.0674,.1431)	(.0000,.0660)
[1.0,1.5)	(.1077,.1694)	(.0650,.1429)	(.0000,.0560)
[1.5,2.0)	(.1308,.1734)	(.0690,.1426)	(.0071,.0601)
[2.0,2.5)	(.1266,.1732)	(.0952,.1472)	(.0074,.0672)
[2.5,3.0)	(.1204,.1409)	(.0983,.1160)	(.0336,.0745)
[3.0,3.5)	(.1090,.1716)	(.0905,.1517)	(.0490,.0787)
[3.5,4.0)	(.0915,.1265)	(.0835,.1145)	(.0530,.0809)
[4.0,4.5)	(.1110,.1281)	(.0995,.1193)	(.0605,.0871)
[4.5,5.0)	(.1040,.1168)	(.1010,.1104)	(.0735,.0900)

NOMINAL ALPHA = .05			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0271,.1431)	(.0175,.1431)	(.0000,.0395)
[1.0,1.5)	(.0439,.1429)	(.0258,.1429)	(.0000,.0245)
[1.5,2.0)	(.0600,.0929)	(.0310,.0606)	(.0000,.0330)
[2.0,2.5)	(.0690,.0981)	(.0386,.0634)	(.0000,.0267)
[2.5,3.0)	(.0645,.0805)	(.0405,.0635)	(.0064,.0305)
[3.0,3.5)	(.0575,.0838)	(.0410,.0670)	(.0120,.0291)
[3.5,4.0)	(.0425,.0680)	(.0375,.0565)	(.0110,.0355)
[4.0,4.5)	(.0585,.0789)	(.0494,.0706)	(.0157,.0361)
[4.5,5.0)	(.0565,.0660)	(.0490,.0580)	(.0260,.0372)

NOMINAL ALPHA = .01			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0000,.0143)	(.0000,.0225)	(.0000,.0090)
[1.0,1.5)	(.0000,.0171)	(.0000,.0132)	(.0000,.0035)
[1.5,2.0)	(.0060,.0193)	(.0028,.0140)	(.0000,.0070)
[2.0,2.5)	(.0074,.0200)	(.0036,.0105)	(.0000,.0046)
[2.5,3.0)	(.0110,.0185)	(.0052,.0135)	(.0000,.0085)
[3.0,3.5)	(.0146,.0240)	(.0063,.0113)	(.0000,.0060)
[3.5,4.0)	(.0105,.0180)	(.0065,.0105)	(.0000,.0040)
[4.0,4.5)	(.0136,.0195)	(.0045,.0118)	(.0000,.0044)
[4.5,5.0)	(.0120,.0170)	(.0088,.0105)	(.0005,.0040)
[4.5,5.0)	(.0120,.0170)	(.0088,.0105)	(.0005,.0040)

MINIMUM AND MAXIMUM SIGNIFICANCE LEVELS
FOR MINIMUM CELL EXPECTATION INTERVALS(MCEI)
TABLE: 2X3

NOMINAL ALPHA = .10			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0675,.1550)	(.0550,.1080)	(.0000,.0525)
[1.0,1.5)	(.1185,.1715)	(.0730,.1115)	(.0000,.0600)
[1.5,2.0)	(.1140,.1720)	(.0830,.1125)	(.0020,.0725)
[2.0,2.5)	(.1195,.1685)	(.0905,.1085)	(.0005,.0670)
[2.5,3.0)	(.1090,.1430)	(.0875,.1075)	(.0360,.0765)
[3.0,3.5)	(.1050,.1515)	(.0935,.1090)	(.0190,.0685)
[3.5,4.0)	(.1045,.1420)	(.0925,.1180)	(.0360,.0875)
[4.0,4.5)	(.1115,.1305)	(.0970,.1110)	(.0370,.0830)
[4.5,5.0)	(.0960,.1315)	(.0895,.1165)	(.0605,.0765)

NOMINAL ALPHA = .05			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0290,.0745)	(.0155,.0605)	(.0000,.0285)
[1.0,1.5)	(.0545,.0890)	(.0305,.0540)	(.0000,.0245)
[1.5,2.0)	(.0540,.0905)	(.0280,.0520)	(.0005,.0265)
[2.0,2.5)	(.0650,.0930)	(.0405,.0550)	(.0000,.0270)
[2.5,3.0)	(.0550,.0770)	(.0400,.0560)	(.0115,.0325)
[3.0,3.5)	(.0560,.0870)	(.0430,.0535)	(.0015,.0240)
[3.5,4.0)	(.0550,.0785)	(.0390,.0580)	(.0100,.0395)
[4.0,4.5)	(.0625,.0680)	(.0430,.0560)	(.0075,.0430)
[4.5,5.0)	(.0460,.0695)	(.0385,.0545)	(.0155,.0295)

NOMINAL ALPHA = .01			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0025,.0165)	(.0015,.0230)	(.0000,.0050)
[1.0,1.5)	(.0070,.0215)	(.0020,.0140)	(.0000,.0020)
[1.5,2.0)	(.0055,.0175)	(.0020,.0115)	(.0000,.0040)
[2.0,2.5)	(.0095,.0255)	(.0025,.0120)	(.0000,.0030)
[2.5,3.0)	(.0090,.0195)	(.0060,.0115)	(.0000,.0040)
[3.0,3.5)	(.0090,.0225)	(.0050,.0110)	(.0000,.0045)
[3.5,4.0)	(.0100,.0230)	(.0055,.0155)	(.0000,.0090)
[4.0,4.5)	(.0140,.0185)	(.0070,.0125)	(.0000,.0055)
[4.5,5.0)	(.0115,.0190)	(.0065,.0105)	(.0010,.0045)
[4.5,5.0)	(.0115,.0190)	(.0065,.0105)	(.0010,.0045)

MINIMUM AND MAXIMUM SIGNIFICANCE LEVELS
FOR MINIMUM CELL EXPECTATION INTERVALS(MCEI)
TABLE: 2X4

NOMINAL ALPHA = .10			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0775,.1610)	(.0590,.1070)	(.0000,.0440)
[1.0,1.5)	(.1070,.1690)	(.0575,.1060)	(.0005,.0390)
[1.5,2.0)	(.1185,.1710)	(.0725,.1055)	(.0000,.0505)
[2.0,2.5)	(.1265,.1765)	(.0770,.1005)	(.0000,.0530)
[2.5,3.0)	(.1185,.1470)	(.0925,.1075)	(.0275,.0595)
[3.0,3.5)	(.1240,.1680)	(.0895,.1110)	(.0065,.0635)
[3.5,4.0)	(.1145,.1330)	(.0925,.1115)	(.0465,.0715)
[4.0,4.5)	(.1245,.1490)	(.0945,.1085)	(.0285,.0705)
[4.5,5.0)	(.1010,.1260)	(.0910,.1055)	(.0420,.0635)

NOMINAL ALPHA = .05			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0250,.0810)	(.0150,.0640)	(.0000,.0235)
[1.0,1.5)	(.0510,.0900)	(.0225,.0505)	(.0000,.0105)
[1.5,2.0)	(.0550,.0940)	(.0285,.0565)	(.0000,.0210)
[2.0,2.5)	(.0660,.1025)	(.0365,.0520)	(.0000,.0190)
[2.5,3.0)	(.0620,.0765)	(.0375,.0510)	(.0045,.0235)
[3.0,3.5)	(.0645,.0900)	(.0375,.0540)	(.0010,.0235)
[3.5,4.0)	(.0555,.0760)	(.0415,.0605)	(.0140,.0275)
[4.0,4.5)	(.0655,.0830)	(.0445,.0540)	(.0060,.0285)
[4.5,5.0)	(.0565,.0705)	(.0455,.0560)	(.0100,.0265)

NOMINAL ALPHA = .01			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0030,.0195)	(.0015,.0235)	(.0000,.0015)
[1.0,1.5)	(.0080,.0205)	(.0015,.0100)	(.0000,.0010)
[1.5,2.0)	(.0070,.0205)	(.0005,.0135)	(.0000,.0025)
[2.0,2.5)	(.0120,.0230)	(.0015,.0120)	(.0000,.0015)
[2.5,3.0)	(.0115,.0195)	(.0040,.0115)	(.0000,.0035)
[3.0,3.5)	(.0125,.0265)	(.0050,.0170)	(.0000,.0040)
[3.5,4.0)	(.0095,.0195)	(.0050,.0115)	(.0000,.0040)
[4.0,4.5)	(.0155,.0255)	(.0055,.0105)	(.0000,.0045)
[4.5,5.0)	(.0075,.0165)	(.0055,.0115)	(.0005,.0035)
[4.5,5.0)	(.0075,.0165)	(.0055,.0115)	(.0005,.0035)

MINIMUM AND MAXIMUM SIGNIFICANCE LEVELS
FOR MINIMUM CELL EXPECTATION INTERVALS(MCEI)

TABLE: 3X3

NOMINAL ALPHA = .10			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0600,.1695)	(.0695,.1150)	(.0000,.0000)
[1.0,1.5)	(.1100,.1765)	(.0840,.1070)	(.0000,.0000)
[1.5,2.0)	(.1295,.1925)	(.0880,.1130)	(.0000,.0000)
[2.0,2.5)	(.1075,.1500)	(.0860,.1150)	(.0000,.0000)
[2.5,3.0)	(.1125,.1720)	(.0925,.1120)	(.0000,.0000)
[3.0,3.5)	(.1155,.1365)	(.0875,.1020)	(.0000,.0000)
[3.5,4.0)	(.1030,.1435)	(.0905,.1120)	(.0000,.0000)
[4.0,4.5)	(.1015,.1350)	(.0890,.1060)	(.0000,.0000)
[4.5,5.0)	(.1170,.1260)	(.0940,.0990)	(.0000,.0000)

NOMINAL ALPHA = .05			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0245,.0920)	(.0335,.0595)	(.0000,.0000)
[1.0,1.5)	(.0445,.0965)	(.0395,.0580)	(.0000,.0000)
[1.5,2.0)	(.0660,.1035)	(.0390,.0545)	(.0000,.0000)
[2.0,2.5)	(.0600,.0920)	(.0375,.0545)	(.0000,.0000)
[2.5,3.0)	(.0595,.1005)	(.0430,.0550)	(.0000,.0000)
[3.0,3.5)	(.0610,.0700)	(.0430,.0455)	(.0000,.0000)
[3.5,4.0)	(.0530,.0795)	(.0410,.0530)	(.0000,.0000)
[4.0,4.5)	(.0490,.0700)	(.0400,.0540)	(.0000,.0000)
[4.5,5.0)	(.0605,.0645)	(.0425,.0480)	(.0000,.0000)

NOMINAL ALPHA = .01			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0035,.0160)	(.0040,.0210)	(.0000,.0000)
[1.0,1.5)	(.0035,.0185)	(.0050,.0175)	(.0000,.0000)
[1.5,2.0)	(.0100,.0205)	(.0035,.0115)	(.0000,.0000)
[2.0,2.5)	(.0100,.0195)	(.0055,.0150)	(.0000,.0000)
[2.5,3.0)	(.0090,.0195)	(.0060,.0100)	(.0000,.0000)
[3.0,3.5)	(.0120,.0150)	(.0050,.0080)	(.0000,.0000)
[3.5,4.0)	(.0100,.0210)	(.0075,.0110)	(.0000,.0000)
[4.0,4.5)	(.0110,.0180)	(.0065,.0125)	(.0000,.0000)
[4.5,5.0)	(.0155,.0185)	(.0080,.0085)	(.0000,.0000)
[4.5,5.0)	(.0155,.0185)	(.0080,.0085)	(.0000,.0000)

MINIMUM AND MAXIMUM SIGNIFICANCE LEVELS
FOR MINIMUM CELL EXPECTATION INTERVALS(MCEI)
TABLE: 2X5

NOMINAL ALPHA = .10			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0750,.1865)	(.0745,.1125)	(.0000,.0230)
[1.0,1.5)	(.1135,.1890)	(.0660,.1095)	(.0000,.0330)
[1.5,2.0)	(.1285,.1855)	(.0740,.1105)	(.0005,.0400)
[2.0,2.5)	(.1210,.1845)	(.0845,.1080)	(.0000,.0415)
[2.5,3.0)	(.1245,.1740)	(.0855,.1025)	(.0105,.0575)
[3.0,3.5)	(.1135,.1590)	(.0875,.1035)	(.0095,.0590)
[3.5,4.0)	(.1155,.1490)	(.0965,.1055)	(.0175,.0650)
[4.0,4.5)	(.1110,.1510)	(.0835,.1080)	(.0265,.0675)
[4.5,5.0)	(.1200,.1270)	(.0995,.1040)	(.0405,.0625)

MINAL ALPHA = .05			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0320,.0950)	(.0305,.0770)	(.0000,.0075)
[1.0,1.5)	(.0485,.0970)	(.0165,.0650)	(.0000,.0155)
[1.5,2.0)	(.0570,.0945)	(.0255,.0525)	(.0000,.0115)
[2.0,2.5)	(.0670,.1020)	(.0330,.0530)	(.0000,.0150)
[2.5,3.0)	(.0660,.0850)	(.0345,.0545)	(.0020,.0235)
[3.0,3.5)	(.0660,.0955)	(.0330,.0550)	(.0005,.0255)
[3.5,4.0)	(.0635,.0840)	(.0410,.0560)	(.0040,.0270)
[4.0,4.5)	(.0580,.0880)	(.0395,.0580)	(.0045,.0295)
[4.5,5.0)	(.0640,.0705)	(.0455,.0500)	(.0100,.0195)

NOMINAL ALPHA = .01			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0025,.0180)	(.0025,.0230)	(.0000,.0005)
[1.0,1.5)	(.0085,.0230)	(.0000,.0220)	(.0000,.0010)
[1.5,2.0)	(.0105,.0230)	(.0020,.0120)	(.0000,.0015)
[2.0,2.5)	(.0125,.0225)	(.0020,.0120)	(.0000,.0020)
[2.5,3.0)	(.0130,.0195)	(.0025,.0090)	(.0000,.0020)
[3.0,3.5)	(.0125,.0220)	(.0050,.0110)	(.0000,.0030)
[3.5,4.0)	(.0135,.0190)	(.0065,.0100)	(.0005,.0020)
[4.0,4.5)	(.0105,.0265)	(.0055,.0105)	(.0000,.0030)
[4.5,5.0)	(.0130,.0150)	(.0070,.0090)	(.0000,.0015)
[4.5,5.0)	(.0130,.0150)	(.0070,.0090)	(.0000,.0015)

MINIMUM AND MAXIMUM SIGNIFICANCE LEVELS
FOR MINIMUM CELL EXPECTATION INTERVALS(MCEI)
TABLE: S2X2X2

NOMINAL ALPHA = .10			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.1390,.2675)	(.1000,.1520)	(.0080,.0310)
[1.0,1.5)	(.1325,.2345)	(.1000,.1455)	(.0085,.0455)
[1.5,2.0)	(.1285,.1790)	(.1020,.1430)	(.0180,.0650)
[2.0,2.5)	(.1260,.1730)	(.1045,.1305)	(.0285,.0730)
[2.5,3.0)	(.1270,.1615)	(.1075,.1225)	(.0385,.0780)
[3.0,3.5)	(.1120,.1480)	(.1005,.1230)	(.0580,.0885)
[3.5,4.0)	(.1225,.1285)	(.1135,.1170)	(.0670,.0940)
[4.0,4.5)	(.1200,.1295)	(.1080,.1245)	(.0775,.1080)
[4.5,5.0)	(.1100,.1205)	(.1080,.1155)	(.0955,.0980)

NOMINAL ALPHA = .05			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0600,.1130)	(.0475,.0755)	(.0005,.0060)
[1.0,1.5)	(.0605,.1060)	(.0455,.0695)	(.0010,.0160)
[1.5,2.0)	(.0660,.0980)	(.0460,.0780)	(.0030,.0250)
[2.0,2.5)	(.0610,.0910)	(.0510,.0660)	(.0035,.0290)
[2.5,3.0)	(.0705,.0870)	(.0510,.0610)	(.0090,.0340)
[3.0,3.5)	(.0630,.0885)	(.0525,.0665)	(.0185,.0360)
[3.5,4.0)	(.0625,.0715)	(.0575,.0585)	(.0255,.0455)
[4.0,4.5)	(.0640,.0715)	(.0505,.0660)	(.0290,.0495)
[4.5,5.0)	(.0515,.0620)	(.0485,.0565)	(.0380,.0435)

NOMINAL ALPHA = .01			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0095,.0255)	(.0060,.0180)	(.0000,.0000)
[1.0,1.5)	(.0105,.0255)	(.0050,.0170)	(.0000,.0015)
[1.5,2.0)	(.0105,.0220)	(.0060,.0150)	(.0000,.0045)
[2.0,2.5)	(.0105,.0240)	(.0070,.0160)	(.0000,.0050)
[2.5,3.0)	(.0135,.0225)	(.0085,.0160)	(.0000,.0040)
[3.0,3.5)	(.0145,.0230)	(.0090,.0165)	(.0000,.0040)
[3.5,4.0)	(.0155,.0200)	(.0090,.0145)	(.0020,.0070)
[4.0,4.5)	(.0150,.0205)	(.0120,.0160)	(.0025,.0100)
[4.5,5.0)	(.0105,.0180)	(.0085,.0135)	(.0045,.0080)
[4.5,5.0)	(.0105,.0180)	(.0085,.0135)	(.0045,.0080)

MINIMUM AND MAXIMUM SIGNIFICANCE LEVELS
FOR MINIMUM CELL EXPECTATION INTERVALS(MCEI)
TABLE: 2X2X2

NOMINAL ALPHA = .10			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0950,.1620)	(.0760,.1040)	(.0005,.0360)
[1.0,1.5)	(.1090,.1655)	(.0835,.1095)	(.0010,.0435)
[1.5,2.0)	(.1180,.1735)	(.0870,.1020)	(.0025,.0475)
[2.0,2.5)	(.1100,.1910)	(.0845,.1120)	(.0030,.0565)
[2.5,3.0)	(.1145,.1265)	(.0850,.0955)	(.0325,.0530)
[3.0,3.5)	(.1100,.1620)	(.0865,.1180)	(.0140,.0670)
[3.5,4.0)	(.1210,.1425)	(.0950,.1075)	(.0470,.0635)
[4.0,4.5)	(.1140,.1450)	(.0950,.1005)	(.0255,.0670)
[4.5,5.0)	(.1120,.1220)	(.0920,.1090)	(.0520,.0775)

NOMINAL ALPHA = .05			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0395,.0835)	(.0330,.0585)	(.0000,.0150)
[1.0,1.5)	(.0530,.0860)	(.0375,.0545)	(.0000,.0180)
[1.5,2.0)	(.0605,.0980)	(.0375,.0580)	(.0005,.0215)
[2.0,2.5)	(.0565,.0965)	(.0425,.0585)	(.0005,.0220)
[2.5,3.0)	(.0580,.0710)	(.0365,.0535)	(.0095,.0235)
[3.0,3.5)	(.0585,.0905)	(.0415,.0620)	(.0050,.0310)
[3.5,4.0)	(.0625,.0775)	(.0490,.0520)	(.0145,.0270)
[4.0,4.5)	(.0550,.0835)	(.0400,.0535)	(.0075,.0300)
[4.5,5.0)	(.0590,.0640)	(.0425,.0520)	(.0160,.0375)

NOMINAL ALPHA = .01			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0050,.0175)	(.0065,.0195)	(.0000,.0015)
[1.0,1.5)	(.0085,.0180)	(.0060,.0135)	(.0000,.0025)
[1.5,2.0)	(.0075,.0195)	(.0055,.0125)	(.0000,.0045)
[2.0,2.5)	(.0100,.0240)	(.0060,.0135)	(.0000,.0030)
[2.5,3.0)	(.0105,.0155)	(.0070,.0100)	(.0005,.0030)
[3.0,3.5)	(.0135,.0250)	(.0080,.0120)	(.0000,.0020)
[3.5,4.0)	(.0120,.0160)	(.0075,.0100)	(.0010,.0035)
[4.0,4.5)	(.0105,.0195)	(.0075,.0125)	(.0000,.0070)
[4.5,5.0)	(.0120,.0140)	(.0060,.0095)	(.0010,.0040)
[4.5,5.0)	(.0120,.0140)	(.0060,.0095)	(.0010,.0040)

MINIMUM AND MAXIMUM SIGNIFICANCE LEVELS
FOR MINIMUM CELL EXPECTATION INTERVALS(MCEI)
TABLE: 2X2X3

NOMINAL ALPHA = .10			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0475,.1150)	(.0885,.1350)	(.0000,.0000)
[1.0,1.5)	(.0670,.1190)	(.0905,.1345)	(.0000,.0000)
[1.5,2.0)	(.0580,.1350)	(.0885,.1265)	(.0000,.0000)
[2.0,2.5)	(.0775,.1385)	(.0930,.1175)	(.0000,.0000)
[2.5,3.0)	(.0860,.1475)	(.0985,.1090)	(.0000,.0000)
[3.0,3.5)	(.0850,.1325)	(.0890,.1060)	(.0000,.0000)
[3.5,4.0)	(.0805,.0970)	(.0785,.0970)	(.0000,.0000)
[4.0,4.5)	(.1025,.1260)	(.0855,.1215)	(.0000,.0000)
[4.5,5.0)	(.1320,.1405)	(.0985,.1010)	(.0000,.0000)

NOMINAL ALPHA = .05			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0195,.0530)	(.0470,.0990)	(.0000,.0000)
[1.0,1.5)	(.0265,.0560)	(.0460,.0835)	(.0000,.0000)
[1.5,2.0)	(.0195,.0735)	(.0455,.0860)	(.0000,.0000)
[2.0,2.5)	(.0360,.0780)	(.0460,.0695)	(.0000,.0000)
[2.5,3.0)	(.0390,.0770)	(.0550,.0670)	(.0000,.0000)
[3.0,3.5)	(.0410,.0665)	(.0445,.0620)	(.0000,.0000)
[3.5,4.0)	(.0355,.0445)	(.0355,.0640)	(.0000,.0000)
[4.0,4.5)	(.0440,.0600)	(.0400,.0755)	(.0000,.0000)
[4.5,5.0)	(.0625,.0680)	(.0455,.0485)	(.0000,.0000)

NOMINAL ALPHA = .01			
MCEI	KULLBACK	PEARSON	GSK
[0.5,1.0)	(.0020,.0080)	(.0130,.0520)	(.0000,.0000)
[1.0,1.5)	(.0015,.0115)	(.0135,.0415)	(.0000,.0000)
[1.5,2.0)	(.0015,.0150)	(.0125,.0380)	(.0000,.0000)
[2.0,2.5)	(.0055,.0145)	(.0115,.0320)	(.0000,.0000)
[2.5,3.0)	(.0055,.0180)	(.0105,.0260)	(.0000,.0000)
[3.0,3.5)	(.0060,.0135)	(.0095,.0210)	(.0000,.0000)
[3.5,4.0)	(.0055,.0095)	(.0100,.0175)	(.0000,.0000)
[4.0,4.5)	(.0055,.0110)	(.0085,.0260)	(.0000,.0000)
[4.5,5.0)	(.0090,.0125)	(.0070,.0115)	(.0000,.0000)
[4.5,5.0)	(.0090,.0125)	(.0070,.0115)	(.0000,.0000)

APPENDIX J

MINIMUM SAMPLE SIZES (N_m)

MINIMUM N (N_m)TABLE: 2×2

VECTOR	$\alpha = .10$			$\alpha = .05$			$\alpha = .01$		
	<u>K</u>	<u>P</u>	<u>G</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>K</u>	<u>P</u>	<u>G</u>
1	>96	>96	>96	>96	>96	>96	>96	>96	>96
2	>96	>96	>96	>96	>96	>96	>96	46	>96
3	>96	46	>96	>96	52	>96	>96	10	>96
4	96	29	>96	96	48	>96	>96	48	>96
5	88	27	>96	96	44	>96	>96	56	>96
6	86	36	>96	96	38	>96	84	21	>96
7	62	<8	80	75	16	96	70	9	90
8	48	<8	69	54	18	96	52	<8	>96
9	45	9	64	48	17	84	48	16	88
10	46	13	57	46	<8	70	40	<8	78
11	35	21	36	38	<8	46	30	<8	60
12	35	22	33	36	<8	44	30	<8	56
13	35	22	28	36	17	32	22	<8	40
14	36	23	26	37	18	30	21	<8	37
15	37	23	25	37	18	23	20	<8	32

MINIMUM N (N_m)TABLE: 2×3

VECTOR	$\alpha = .10$			$\alpha = .05$			$\alpha = .01$		
	<u>K</u>	<u>P</u>	<u>G</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>K</u>	<u>P</u>	<u>G</u>
1	>96	>96	>96	>96	62	>96	>96	>96	>96
2	>96	17	>96	>96	33	>96	>96	81	>96
3	>96	>96	>96	>96	25	>96	>96	24	>96
4	>96	36	>96	>96	48	>96	>96	13	>96
5	>96	14	>96	>96	<12	>96	>96	30	>96
6	>96	<12	>96	>96	<12	>96	>96	<12	>96
7	>96	<12	>96	>96	19	>96	36	<12	>96
8	82	<12	>96	96	25	>96	89	13	>96
9	>96	30	>96	>96	26	>96	>96	>96	>96
10	>96	<12	>96	>96	<12	>96	45	<12	>96
11	90	22	>96	96	20	>96	50	<12	>96
12	64	<12	94	80	<12	93	75	<12	>96
13	>96	30	>96	>96	40	>96	84	30	>96
14	74	<12	>96	69	18	>96	50	21	>96
15	50	<12	77	48	<12	>96	36	20	80
16	48	<12	60	45	<12	70	35	15	>96
17	>96	32	>96	>96	45	>96	>96	61	>96
18	87	24	>96	90	26	>96	82	32	>96
19	70	<12	84	62	18	>96	67	24	>96
20	74	<12	58	55	<12	72	44	22	74
21	49	<12	52	38	<12	60	36	15	65

MINIMUM N (N_m)TABLE: 2×4

VECTOR	$\alpha = .10$			$\alpha = .05$			$\alpha = .01$		
	<u>K</u>	<u>P</u>	<u>G</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>K</u>	<u>P</u>	<u>G</u>
1	>96	37	>96	>96	76	>96	>96	>96	>96
2	>96	21	>96	>96	24	>96	>96	94	>96
3	>96	22	>96	>96	23	>96	>96	80	>96
4	>96	50	>96	>96	50	>96	>96	<16	>96
5	>96	16	>96	>96	<16	>96	>96	25	>96
6	>96	<16	>96	>96	<16	>96	>96	32	>96
7	>96	<16	>96	>96	<16	>96	>96	<16	>96
8	>96	<16	>96	>96	<16	>96	>96	<16	>96
9	>96	40	>96	>96	28	>96	>96	>96	>96
10	>96	24	>96	>96	16	>96	50	20	>96
11	>96	18	>96	>96	18	>96	40	<16	>96
12	>96	16	>96	96	<16	>96	82	<16	>96
13	>96	40	>96	>96	44	>96	78	80	>96
14	>96	16	>96	>96	70	>96	>96	55	>96
15	>96	24	>96	>96	28	>96	72	40	>96
16	74	<16	>96	82	30	>96	80	30	>96
17	>96	47	>96	>96	50	>96	>96	68	>96
18	>96	35	>96	>96	38	>96	85	55	>96
19	88	19	>96	96	28	>96	82	37	>96
20	58	<16	>96	79	21	>96	70	23	>96
21	70	<16	80	80	<16	>96	70	24	>96

MINIMUM N (N_m)

TABLE: 3 x 3

<u>VECTOR</u>	<u>$\alpha = .10$</u>		<u>$\alpha = .05$</u>		<u>$\alpha = .01$</u>	
	<u>K</u>	<u>P</u>	<u>K</u>	<u>P</u>	<u>K</u>	<u>P</u>
1	>108	48	>108	60	>108	>108
2	>108	3]	>108	54	>108	99
3	>108	20	>108	27	>108	79
4	>108	50	>108	32	>108	<18
5	>108	26	>108	49	>108	91
6	>108	<18	>108	<18	>108	52
7	>108	<18	>108	22	>108	18
8	>108	<18	>108	30	>108	30
9	>108	<18	>108	<18	>108	46
10	>108	<18	>108	<18	>108	<18
11	>108	<18	>108	<18	>108	20
12	>108	<18	>108	<18	90	27
13	>108	<18	>108	<18	>108	<18
14	>108	<18	>108	20	78	<18
15	>108	<18	>108	<18	>108	<18
16	>108	<18	>108	20	>108	20
17	90	<18	72	<18	74	<18
18	80	<18	74	<18	72	<18
19	80	<18	80	23	72	25
20	80	<18	64	<18	67	22
21	63	<18	51	18	51	<18

MINIMUM N (N_m)
TABLE: 2×5

VECTOR	$\alpha = .10$			$\alpha = .05$			$\alpha = .01$		
	<u>K</u>	<u>P</u>	<u>G</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>K</u>	<u>P</u>	<u>G</u>
1	>100	50	>100	>100	74	>100	>100	>100	>100
2	>100	31	>100	>100	42	>100	>100	85	>100
3	>100	24	>100	>100	44	>100	>100	54	>100
4	>100	<20	>100	>100	55	>100	>100	55	>100
5	>100	<20	>100	>100	<20	>100	>100	<20	>100
6	>100	<20	>100	>100	50	>100	>100	<20	>100
7	>100	55	>100	>100	58	>100	>100	46	>100
8	>100	<20	>100	>100	43	>100	>100	56	>100
9	>100	<20	>100	>100	40	>100	>100	33	>100
10	>100	27	>100	>100	45	>100	>100	44	>100
11	>100	27	>100	>100	31	>100	94	40	>100
12	95	21	>100	97	24	>100	85	28	>100
13	>100	39	>100	>100	60	>100	>100	52	>100
14	>100	24	>100	>100	60	>100	96	40	>100
15	>100	23	>100	>100	27	>100	86	32	>100
16	79	<20	>100	85	25	>100	60	26	>100

MINIMUM N (N_m)
TABLE: $S2 \times 2 \times 2$

<u>VECTOR</u>	<u>$\alpha = .10$</u>			<u>$\alpha = .05$</u>			<u>$\alpha = .01$</u>		
	<u>K</u>	<u>P</u>	<u>G</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>K</u>	<u>P</u>	<u>G</u>
1	>104	>104	>104	>104	>104	>104	>104	>104	>104
2	>104	>104	>104	>104	>104	>104	>104	>104	>104
3	>104	>104	>104	>104	>104	>104	>104	>104	>104
4	>104	>104	>104	>104	>104	>104	>104	>104	>104
5	>104	>104	>104	>104	85	>104	>104	>104	>104
6	>104	>104	>104	>104	>104	>104	>104	>104	>104
7	>104	>104	>104	>104	>104	>104	>104	>104	>104
8	>104	>104	>104	>104	60	>104	>104	>104	>104
9	>104	>104	99	>104	48	>104	>104	>104	>104
10	>104	93	>104	>104	35	>104	>104	>104	>104
11	>104	>104	>104	>104	36	>104	>104	>104	>104
12	>104	>104	89	>104	42	>104	>104	>104	98
13	71	56	80	99	57	91	74	>104	85
14	99	44	55	94	36	68	81	72	91
15	74	56	51	60	49	88	64	56	79
16	54	47	40	54	42	50	59	40	50
17	58	38	38	55	40	32	51	37	36

MINIMUM N (N_m)
TABLE: $2 \times 2 \times 2$

VECTOR	$\alpha = .10$			$\alpha = .05$			$\alpha = .01$		
	<u>K</u>	<u>P</u>	<u>G</u>	<u>K</u>	<u>P</u>	<u>G</u>	<u>K</u>	<u>P</u>	<u>G</u>
1	>96	>96	>96	>96	>96	>96	>96	>96	>96
2	>96	63	>96	>96	>96	>96	>96	>96	>96
3	>96	56	>96	>96	73	>96	>96	>96	>96
4	>96	37	>96	>96	88	>96	>96	>96	>96
5	>96	27	>96	>96	31	>96	>96	63	>96
6	>96	<16	>96	>96	23	>96	>96	46	>96
7	>96	<16	>96	>96	<16	>96	>96	<16	>96
8	>96	37	>96	>96	34	>96	>96	20	>96
9	>96	43	>96	>96	56	>96	>96	17	>96
10	>96	19	>96	>96	35	>96	>96	>96	>96
11	>96	<16	>96	>96	<16	>96	>96	41	>96
12	>96	<16	>96	>96	<16	>96	>96	42	>96
13	>96	<16	>96	>96	<16	>96	83	<16	>96
14	>96	<16	>96	>96	<16	>96	>96	<16	>96
15	96	<16	>96	>96	<16	>96	68	<16	>96
16	>96	<16	>96	72	<16	>96	56	<16	>96
17	96	<16	>96	>96	<16	>96	>96	<16	>96
18	>96	<16	>96	>96	<16	>96	64	<16	>96
19	63	<16	>96	63	16	>96	61	<16	>96
20	66	<16	>96	49	<16	>96	63	<16	>96
21	63	<16	>96	65	<16	>96	45	<16	>96
22	56	<16	96	48	<16	>96	45	<16	96

MINIMUM N (N_m)
TABLE: $2 \times 2 \times 3$

VECTOR	$\alpha = .10$		$\alpha = .05$		$\alpha = .01$	
	<u>K</u>	<u>P</u>	<u>K</u>	<u>P</u>	<u>K</u>	<u>P</u>
1	>96	>96	>96	>96	>96	>96
2	>96	96	>96	>96	>96	>96
3	>96	>96	>96	>96	>96	>96
4	>96	42	>96	71	>96	96
5	>96	50	>96	63	>96	>96
6	>96	<24	>96	25	>96	63
7	>96	<24	>96	49	>96	96
8	>96	<24	>96	<24	>96	34
9	>96	96	>96	>96	>96	>96
10	>96	<24	>96	26	>96	42
11	>96	43	>96	76	>96	96
12	>96	25	>96	26	>96	84
13	>96	<24	>96	36	>96	78
14	>96	<24	>96	<24	>96	<24
15	>96	27	>96	<24	>96	76
16	>96	<24	>96	46	>96	25
17	>96	<24	>96	31	>96	92
18	>96	<24	>96	<24	>96	<24
19	>96	<24	>96	<24	>96	36
20	>96	<24	>96	<24	>96	<24
21	>96	<24	>96	34	>96	<24
22	>96	<24	>96	29	88	<24
23	>96	<24	>96	<24	>96	<24
24	>96	<24	>96	<24	89	<24
25	>96	32	>96	55	>96	76
26	>96	<24	>96	<24	>96	<24
27	>96	44	>96	41	>96	25
28	>96	<24	>96	<24	67	<24
29	>96	<24	>96	<24	>96	41
30	>96	<24	>96	29	>96	<24
31	96	<24	96	<24	69	<24

APPENDIX K

CRITICAL EXPECTED VALUE (CEV) DISTRIBUTIONS

CRITICAL EXPECTED VALUE DISTRIBUTION

TABLE: 2X2
 STATISTIC:GSK
 ALPHA: .10

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.06,.14,.24,.56)	.0361	80	1	1	1	1	1	1	0	0	0	0
(.08,.12,.32,.48)	.0259	69	2	2	1	1	1	0	0	0	0	0
(.10,.10,.40,.40)	.0225	64	2	2	2	2	0	0	0	0	0	0
(.09,.21,.21,.49)	.0216	57	1	1	1	1	1	0	0	0	0	0
(.12,.18,.28,.42)	.0129	36	2	2	2	2	1	1	0	0	0	0
(.15,.15,.35,.35)	.0100	33	2	2	2	2	2	2	0	0	0	0
(.16,.24,.24,.36)	.0051	28	3	3	3	3	1	1	0	0	0	0
(.20,.20,.30,.30)	.0025	26	4	4	4	2	2	0	0	0	0	0
(.25,.25,.25,.25)	.0000	25	4	4	4	4	0	0	0	0	0	0
(.01,.09,.09,.81)	.1056	> 96	3	3	1	1	1	1	1	1	1	1
(.02,.08,.18,.72)	.0769	> 96	2	2	2	1	1	1	1	1	1	0
(.03,.07,.27,.63)	.0564	> 96	2	2	2	2	1	1	1	1	0	0
(.04,.16,.16,.64)	.0531	> 96	1	1	1	1	1	1	1	0	0	0
(.04,.06,.36,.54)	.0441	> 96	2	2	2	2	2	1	1	0	0	0
(.05,.05,.45,.45)	.0400	> 96	2	2	2	2	2	2	0	0	0	0

CRITICAL EXPECTED VALUE DISTRIBUTION

TABLE: 2X3
 STATISTIC: GSK
 ALPHA: .10

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.10,.10,.30,.10,.10,.30)	.0089	84	4	4	0	0	0	0	0	0	0	0
(.06,.12,.12,.14,.28,.28)	.0070	94	1	1	1	1	1	0	0	0	0	0
(.08,.12,.20,.12,.18,.30)	.0052	77	3	1	1	1	0	0	0	0	0	0
(.10,.20,.20,.10,.20,.20)	.0022	58	2	2	2	2	2	0	0	0	0	0
(.12,.12,.16,.18,.18,.24)	.0017	60	3	2	2	0	0	0	0	0	0	0
(.17,.17,.17,.17,.17,.17)	.0000	52	6	6	0	0	0	0	0	0	0	0
(.01,.01,.08,.09,.09,.72)	.0624	> 96	5	5	3	2	2	2	2	2	2	2
(.03,.03,.24,.07,.07,.56)	.0360	> 96	4	4	4	4	2	2	2	2	0	0
(.01,.03,.06,.09,.27,.54)	.0351	> 96	4	4	3	3	3	2	2	2	1	1
(.02,.04,.14,.08,.16,.56)	.0334	> 96	3	3	3	2	2	2	2	1	1	0
(.02,.02,.06,.18,.18,.54)	.0324	> 96	3	3	3	3	3	2	2	2	2	0
(.05,.05,.40,.05,.05,.40)	.0272	> 96	4	4	4	4	4	4	0	0	0	0
(.02,.04,.04,.18,.36,.36)	.0214	> 96	3	3	3	3	3	3	3	1	1	0
(.02,.08,.10,.08,.32,.40)	.0198	> 96	4	3	3	1	1	1	1	1	1	0
(.04,.08,.28,.06,.12,.42)	.0190	> 96	3	3	3	2	2	1	1	0	0	0
(.03,.09,.18,.07,.21,.42)	.0167	> 96	3	3	2	2	1	1	1	1	0	0
(.04,.06,.10,.16,.24,.40)	.0153	> 96	3	2	2	2	2	1	1	0	0	0
(.06,.06,.18,.14,.14,.42)	.0148	> 96	2	2	2	2	2	0	0	0	0	0
(.06,.06,.08,.24,.24,.32)	.0108	> 96	3	3	3	2	2	0	0	0	0	0
(.05,.15,.30,.05,.15,.30)	.0106	> 96	2	2	2	2	2	2	0	0	0	0
(.04,.16,.20,.06,.24,.30)	.0086	> 96	2	2	2	2	2	1	1	0	0	0

TABLE: 2X4
 STATISTIC: GSK
 ALPHA: .10

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.13,.13,.13,.13,.13,.13,.13)	.0000	80	0	0	0	0	0	0	0	0	0	0
(.01,.01,.01,.07,.09,.09,.09,.63)	.0377	> 96	7	7	4	4	3	3	3	3	3	3
(.03,.03,.03,.21,.07,.07,.07,.49)	.0221	> 96	6	6	6	6	3	3	3	3	0	0
(.01,.01,.03,.05,.09,.09,.27,.45)	.0213	> 96	6	6	4	4	4	4	3	3	2	2
(.02,.02,.04,.12,.08,.08,.16,.48)	.0201	> 96	5	5	5	3	3	3	3	2	2	0
(.01,.02,.02,.05,.09,.18,.18,.45)	.0192	> 96	5	5	4	4	4	4	3	3	3	1
(.05,.05,.05,.35,.05,.05,.05,.35)	.0169	> 96	6	6	6	6	6	6	0	0	0	0
(.02,.02,.08,.08,.08,.08,.32,.32)	.0133	> 96	6	6	6	2	2	2	2	2	2	0
(.02,.02,.02,.04,.18,.18,.18,.36)	.0131	> 96	4	4	4	4	4	4	4	3	3	0
(.04,.04,.08,.24,.06,.06,.12,.36)	.0117	> 96	5	5	5	4	4	2	2	0	0	0
(.03,.03,.09,.15,.07,.07,.21,.35)	.0105	> 96	5	5	4	4	2	2	2	2	0	0
(.02,.04,.06,.08,.08,.16,.24,.32)	.0099	> 96	5	5	5	3	3	2	2	1	1	0
(.03,.06,.06,.15,.07,.14,.14,.35)	.0090	> 96	4	4	4	4	3	1	1	1	1	0
(.05,.05,.15,.25,.05,.05,.15,.25)	.0069	> 96	4	4	4	4	4	4	0	0	0	0
(.04,.04,.16,.16,.06,.06,.24,.24)	.0065	> 96	4	4	4	4	4	2	2	0	0	0
(.04,.04,.06,.06,.16,.16,.24,.24)	.0065	> 96	4	4	4	4	4	2	2	0	0	0
(.05,.10,.10,.25,.05,.10,.10,.25)	.0056	> 96	6	2	2	2	2	2	0	0	0	0
(.06,.06,.06,.12,.14,.14,.14,.28)	.0047	> 96	3	3	3	3	3	0	0	0	0	0
(.04,.08,.12,.16,.06,.12,.18,.24)	.0039	> 96	3	3	3	2	2	1	1	0	0	0
(.10,.10,.10,.20,.10,.10,.10,.20)	.0019	> 96	6	0	0	0	0	0	0	0	0	0
(.08,.08,.12,.12,.12,.12,.18,.18)	.0013	> 96	2	2	2	0	0	0	0	0	0	0

CRITICAL EXPECTED VALUE DISTRIBUTION

TABLE: 2X2
 STATISTIC: GSK
 ALPHA: .05

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.06,.14,.24,.56)	.0361	96	1	1	1	1	1	0	0	0	0	0
(.08,.12,.32,.48)	.0259	96	1	1	1	0	0	0	0	0	0	0
(.10,.10,.40,.40)	.0225	84	2	2	0	0	0	0	0	0	0	0
(.09,.21,.21,.49)	.0216	70	1	1	1	1	0	0	0	0	0	0
(.12,.18,.28,.42)	.0129	46	2	2	1	1	1	0	0	0	0	0
(.15,.15,.35,.35)	.0100	44	2	2	2	2	0	0	0	0	0	0
(.16,.24,.24,.36)	.0051	32	3	3	3	1	1	0	0	0	0	0
(.20,.20,.30,.30)	.0025	30	4	2	2	2	0	0	0	0	0	0
(.25,.25,.25,.25)	.0000	23	4	4	4	4	4	0	0	0	0	0
(.01,.09,.09,.81)	.1056	> 96	3	3	1	1	1	1	1	1	1	1
(.02,.08,.18,.72)	.0769	> 96	2	2	2	1	1	1	1	1	1	0
(.03,.07,.27,.63)	.0564	> 96	2	2	2	2	1	1	1	1	0	0
(.04,.16,.16,.64)	.0531	> 96	1	1	1	1	1	1	1	0	0	0
(.04,.06,.36,.54)	.0441	> 96	2	2	2	2	2	1	1	0	0	0
(.05,.05,.45,.45)	.0400	> 96	2	2	2	2	2	2	0	0	0	0

TABLE: 2X3
 STATISTIC:GSK
 ALPHA: .05

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.06,.12,.12,.14,.28,.28)	.0070	93	1	1	1	1	1	0	0	0	0	0
(.10,.20,.20,.10,.20,.20)	.0022	72	2	2	2	0	0	0	0	0	0	0
(.12,.12,.16,.18,.18,.24)	.0017	70	2	2	0	0	0	0	0	0	0	0
(.17,.17,.17,.17,.17,.17)	.0000	60	0	0	0	0	0	0	0	0	0	0
(.01,.01,.08,.09,.09,.72)	.0624	> 96	5	5	3	2	2	2	2	2	2	2
(.03,.03,.24,.07,.07,.56)	.0360	> 96	4	4	4	4	2	2	2	2	0	0
(.01,.03,.06,.09,.27,.54)	.0351	> 96	4	4	3	3	3	2	2	2	1	1
(.02,.04,.14,.08,.16,.56)	.0334	> 96	3	3	3	2	2	2	2	1	1	0
(.02,.02,.06,.18,.18,.54)	.0324	> 96	3	3	3	3	3	2	2	2	2	0
(.05,.05,.40,.05,.05,.40)	.0272	> 96	4	4	4	4	4	4	0	0	0	0
(.02,.04,.04,.18,.36,.36)	.0214	> 96	3	3	3	3	3	3	3	1	1	0
(.02,.08,.10,.08,.32,.40)	.0198	> 96	4	3	3	1	1	1	1	1	1	0
(.04,.08,.28,.06,.12,.42)	.0190	> 96	3	3	3	2	2	1	1	0	0	0
(.03,.09,.18,.07,.21,.42)	.0167	> 96	3	3	2	2	1	1	1	1	0	0
(.04,.06,.10,.16,.24,.40)	.0153	> 96	3	2	2	2	2	1	1	0	0	0
(.06,.06,.18,.14,.14,.42)	.0148	> 96	2	2	2	2	2	0	0	0	0	0
(.06,.06,.08,.24,.24,.32)	.0108	> 96	3	3	3	2	2	0	0	0	0	0
(.05,.15,.30,.05,.15,.30)	.0106	> 96	2	2	2	2	2	2	0	0	0	0
(.10,.10,.30,.10,.10,.30)	.0089	> 96	4	0	0	0	0	0	0	0	0	0
(.04,.16,.20,.06,.24,.30)	.0086	> 96	2	2	2	2	2	1	1	0	0	0
(.08,.12,.20,.12,.18,.30)	.0052	> 96	1	1	1	0	0	0	0	0	0	0

CRITICAL EXPECTED VALUE DISTRIBUTION

TABLE: 2X2
 STATISTIC:GSK
 ALPHA: .01

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.06,.14,.24,.56)	.0361	90	1	1	1	1	1	0	0	0	0	0
(.10,.10,.40,.40)	.0225	88	2	2	0	0	0	0	0	0	0	0
(.09,.21,.21,.49)	.0216	78	1	1	1	0	0	0	0	0	0	0
(.12,.18,.28,.42)	.0129	60	1	1	1	0	0	0	0	0	0	0
(.15,.15,.35,.35)	.0100	56	2	2	0	0	0	0	0	0	0	0
(.16,.24,.24,.36)	.0051	40	3	1	1	1	0	0	0	0	0	0
(.20,.20,.30,.30)	.0025	37	2	2	2	0	0	0	0	0	0	0
(.25,.25,.25,.25)	.0000	32	4	4	0	0	0	0	0	0	0	0
(.01,.09,.09,.81)	.1056	> 96	3	3	1	1	1	1	1	1	1	1
(.02,.08,.18,.72)	.0769	> 96	2	2	2	1	1	1	1	1	1	0
(.03,.07,.27,.63)	.0564	> 96	2	2	2	2	1	1	1	1	0	0
(.04,.16,.16,.64)	.0531	> 96	1	1	1	1	1	1	1	0	0	0
(.04,.06,.36,.54)	.0441	> 96	2	2	2	2	2	1	1	0	0	0
(.05,.05,.45,.45)	.0400	> 96	2	2	2	2	2	2	0	0	0	0
(.08,.12,.32,.48)	.0259	> 96	1	1	1	0	0	0	0	0	0	0
(.08,.12,.32,.48)	.0259	> 96	1	1	1	0	0	0	0	0	0	0

UNCLASSIFIED

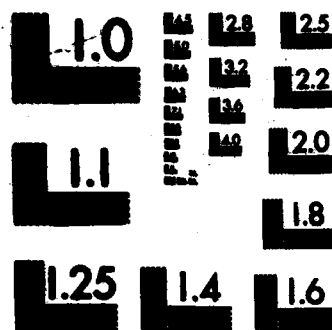
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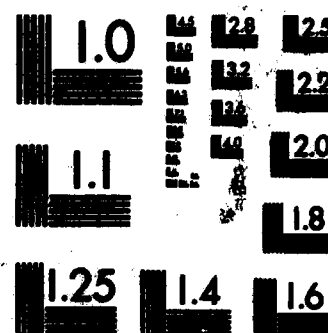
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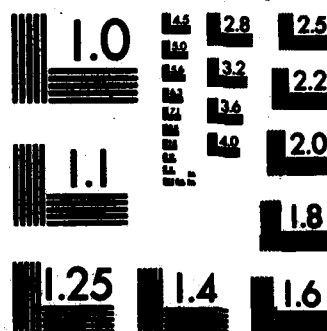
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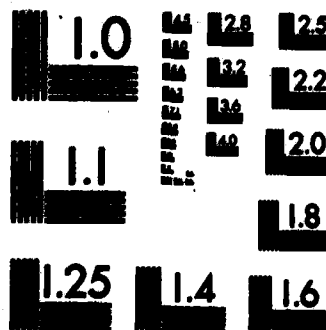
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A



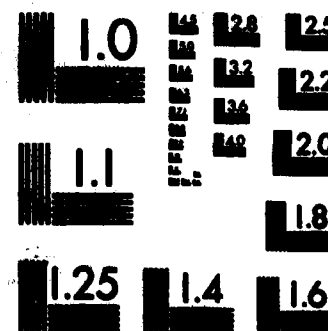
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

TABLE: S2X2X2
 STATISTIC: GSK
 ALPHA: .10

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.032,.048,.048,.072,.128,.192,.192,.288)	.0074	99	4	4	4	3	3	3	1	0	0	0
(.045,.045,.105,.105,.105,.105,.245,.245)	.0054	89	6	2	2	2	2	2	0	0	0	0
(.048,.072,.072,.108,.112,.168,.168,.252)	.0040	80	5	5	3	3	3	1	1	0	0	0
(.075,.075,.075,.075,.175,.175,.175,.175)	.0025	65	4	4	4	4	4	4	0	0	0	0
(.064,.096,.096,.144,.096,.144,.144,.216)	.0020	51	7	7	7	4	4	4	1	0	0	0
(.100,.100,.100,.100,.150,.150,.150,.150)	.0006	40	8	8	8	8	4	4	0	0	0	0
(.125,.125,.125,.125,.125,.125,.125,.125)	.0000	38	8	8	8	8	8	8	0	0	0	0
(.008,.012,.032,.048,.072,.108,.288,.432)	.0206	>104	5	5	5	4	4	4	3	2	2	1
(.009,.021,.021,.049,.081,.189,.189,.441)	.0189	>104	5	5	4	4	4	3	3	3	1	1
(.016,.024,.064,.096,.064,.096,.256,.384)	.0144	>104	6	4	4	4	2	2	2	2	1	0
(.015,.015,.035,.035,.135,.135,.315,.315)	.0141	>104	4	4	4	4	4	4	4	2	2	0
(.018,.042,.042,.098,.072,.168,.168,.392)	.0130	>104	4	4	4	3	3	3	1	1	1	0
(.016,.024,.024,.036,.144,.216,.216,.324)	.0121	>104	4	4	4	4	4	4	4	3	1	0
(.025,.025,.025,.025,.225,.225,.225,.225)	.0100	>104	4	4	4	4	4	4	4	4	0	0
(.030,.030,.070,.070,.120,.120,.280,.280)	.0090	>104	4	4	4	2	2	2	2	0	0	0
(.027,.063,.063,.147,.063,.147,.147,.343)	.0088	>104	4	4	4	4	1	1	1	1	0	0
(.050,.050,.050,.050,.200,.200,.200,.200)	.0056	>104	4	4	4	4	4	0	0	0	0	0

TABLE: S2X2X2
 STATISTIC:GSK
 ALPHA: .05

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.048,.072,.072,.108,.112,.168,.168,.252)	.0040	91	4	3	3	3	1	1	0	0	0	0
(.075,.075,.075,.075,.175,.175,.175,.175)	.0025	68	4	4	4	4	4	0	0	0	0	0
(.064,.096,.096,.144,.096,.144,.144,.216)	.0020	88	4	4	1	1	1	0	0	0	0	0
(.100,.100,.100,.100,.150,.150,.150,.150)	.0006	50	8	8	8	4	4	0	0	0	0	0
(.125,.125,.125,.125,.125,.125,.125,.125)	.0000	32	8	8	8	8	8	8	0	0	0	0
(.008,.012,.032,.048,.072,.108,.288,.432)	.0206	>104	5	5	5	4	4	4	3	2	2	1
(.009,.021,.021,.049,.081,.189,.189,.441)	.0189	>104	5	5	4	4	4	3	3	3	1	1
(.016,.024,.064,.096,.064,.096,.256,.384)	.0144	>104	6	4	4	4	2	2	2	2	1	0
(.015,.015,.035,.035,.135,.135,.315,.315)	.0141	>104	4	4	4	4	4	4	4	2	2	0
(.018,.042,.042,.098,.072,.168,.168,.392)	.0130	>104	4	4	4	3	3	3	1	1	1	0
(.016,.024,.024,.036,.144,.216,.216,.324)	.0121	>104	4	4	4	4	4	4	4	3	1	0
(.025,.025,.025,.025,.225,.225,.225,.225)	.0100	>104	4	4	4	4	4	4	4	4	0	0
(.030,.030,.070,.070,.120,.120,.280,.280)	.0090	>104	4	4	4	2	2	2	2	0	0	0
(.027,.063,.063,.147,.063,.147,.147,.343)	.0088	>104	4	4	4	4	1	1	1	1	0	0
(.032,.048,.048,.072,.128,.192,.192,.288)	.0074	>104	4	4	4	3	3	3	1	0	0	0
(.050,.050,.050,.050,.200,.200,.200,.200)	.0056	>104	4	4	4	4	4	0	0	0	0	0
(.045,.045,.105,.105,.105,.105,.245,.245)	.0054	>104	2	2	2	2	2	2	0	0	0	0

TABLE: S2X2X2
 STATISTIC: GSK
 ALPHA: .01

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.045,.045,.105,.105,.105,.105,.245,.245)	.0054	98	2	2	2	2	2	2	0	0	0	0
(.048,.072,.072,.108,.112,.168,.168,.252)	.0040	85	5	3	3	3	1	1	0	0	0	0
(.075,.075,.075,.075,.175,.175,.175,.175)	.0025	91	4	4	4	4	0	0	0	0	0	0
(.064,.096,.096,.144,.096,.144,.144,.216)	.0020	79	4	4	4	1	1	0	0	0	0	0
(.100,.100,.100,.100,.150,.150,.150,.150)	.0006	50	8	8	8	4	4	0	0	0	0	0
(.125,.125,.125,.125,.125,.125,.125,.125)	.0000	36	8	8	8	8	8	8	0	0	0	0
(.008,.012,.032,.048,.072,.108,.288,.432)	.0206	>104	5	5	5	4	4	4	3	2	2	1
(.009,.021,.021,.049,.081,.189,.189,.441)	.0189	>104	5	5	4	4	4	3	3	3	1	1
(.016,.024,.064,.096,.064,.096,.256,.384)	.0144	>104	6	4	4	4	2	2	2	2	1	0
(.015,.015,.035,.035,.135,.135,.315,.315)	.0141	>104	4	4	4	4	4	4	4	2	2	0
(.018,.042,.042,.098,.072,.168,.168,.392)	.0130	>104	4	4	4	3	3	3	1	1	1	0
(.016,.024,.024,.036,.144,.216,.216,.324)	.0121	>104	4	4	4	4	4	4	4	3	1	0
(.025,.025,.025,.025,.225,.225,.225,.225)	.0100	>104	4	4	4	4	4	4	4	4	0	0
(.030,.030,.070,.070,.120,.120,.280,.280)	.0090	>104	4	4	4	2	2	2	2	0	0	0
(.027,.063,.063,.147,.063,.147,.147,.343)	.0088	>104	4	4	4	4	1	1	1	1	0	0
(.032,.048,.048,.072,.128,.192,.192,.288)	.0074	>104	4	4	4	3	3	3	1	0	0	0
(.050,.050,.050,.050,.200,.200,.200,.200)	.0056	>104	4	4	4	4	4	0	0	0	0	0
(.050,.050,.050,.050,.200,.200,.200,.200)	.0056	>104	4	4	4	4	4	0	0	0	0	0

CRITICAL EXPECTED VALUE DISTRIBUTION

TABLE: 2X2
 STATISTIC: KULLBACK
 ALPHA: .10

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.04,.16,.16,.64)	.0531	86	1	1	1	1	1	1	1	0	0	0
(.04,.06,.36,.54)	.0441	96	2	2	2	2	2	1	1	0	0	0
(.05,.05,.45,.45)	.0400	88	2	2	2	2	2	2	0	0	0	0
(.06,.14,.24,.56)	.0361	62	2	2	1	1	1	1	1	0	0	0
(.08,.12,.32,.48)	.0259	48	2	2	2	2	2	1	1	0	0	0
(.10,.10,.40,.40)	.0225	45	2	2	2	2	2	2	0	0	0	0
(.09,.21,.21,.49)	.0216	46	3	1	1	1	1	1	0	0	0	0
(.12,.18,.28,.42)	.0129	35	3	2	2	2	1	1	0	0	0	0
(.15,.15,.35,.35)	.0100	35	2	2	2	2	2	0	0	0	0	0
(.16,.24,.24,.36)	.0051	35	3	3	1	1	1	0	0	0	0	0
(.20,.20,.30,.30)	.0025	36	2	2	2	0	0	0	0	0	0	0
(.25,.25,.25,.25)	.0000	37	4	0	0	0	0	0	0	0	0	0
(.01,.09,.09,.81)	.1056	> 96	3	3	1	1	1	1	1	1	1	1
(.02,.08,.18,.72)	.0769	> 96	2	2	2	1	1	1	1	1	1	0
(.03,.07,.27,.63)	.0564	> 96	2	2	2	2	1	1	1	1	0	0

TABLE: 2X3
 STATISTIC: KULLBACK
 ALPHA: .10

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.06, .06, .18, .14, .14, .42)	.0148	90	2	2	2	2	2	0	0	0	0	0
(.06, .06, .08, .24, .24, .32)	.0108	82	3	3	3	3	2	2	0	0	0	0
(.05, .15, .30, .05, .15, .30)	.0106	87	2	2	2	2	2	2	0	0	0	0
(.10, .10, .30, .10, .10, .30)	.0089	70	4	4	4	0	0	0	0	0	0	0
(.04, .16, .20, .06, .24, .30)	.0086	74	2	2	2	2	2	2	1	1	0	0
(.06, .12, .12, .14, .28, .28)	.0070	64	4	4	3	1	1	1	1	0	0	0
(.08, .12, .20, .12, .18, .30)	.0052	50	4	3	3	3	1	1	0	0	0	0
(.10, .20, .20, .10, .20, .20)	.0022	74	2	2	2	0	0	0	0	0	0	0
(.12, .12, .16, .18, .18, .24)	.0017	48	5	5	3	2	2	0	0	0	0	0
(.17, .17, .17, .17, .17, .17)	.0000	49	6	6	0	0	0	0	0	0	0	0
(.01, .01, .08, .09, .09, .72)	.0624	> 96	5	5	3	2	2	2	2	2	2	2
(.03, .03, .24, .07, .07, .56)	.0360	> 96	4	4	4	4	2	2	2	2	0	0
(.01, .03, .06, .09, .27, .54)	.0351	> 96	4	4	3	3	3	2	2	2	1	1
(.02, .04, .14, .08, .16, .56)	.0334	> 96	3	3	3	2	2	2	2	1	1	0
(.02, .02, .06, .18, .18, .54)	.0324	> 96	3	3	3	3	3	2	2	2	2	0
(.05, .05, .40, .05, .05, .40)	.0272	> 96	4	4	4	4	4	4	0	0	0	0
(.02, .04, .04, .18, .36, .36)	.0214	> 96	3	3	3	3	3	3	3	1	1	0
(.02, .08, .10, .08, .32, .40)	.0198	> 96	4	3	3	1	1	1	1	1	1	0
(.04, .08, .28, .06, .12, .42)	.0190	> 96	3	3	3	2	2	1	1	0	0	0
(.03, .09, .18, .07, .21, .42)	.0167	> 96	3	3	2	2	1	1	1	1	0	0
(.04, .06, .10, .16, .24, .40)	.0153	> 96	3	2	2	2	2	1	1	0	0	0

TABLE: 2X4
 STATISTIC: KULLBACK
 ALPHA: .10

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.05, .10, .10, .25, .05, .10, .10, .25)	.0056	88	6	6	2	2	2	2	0	0	0	0
(.10, .10, .10, .20, .10, .10, .10, .20)	.0019	58	6	6	6	6	6	0	0	0	0	0
(.08, .08, .12, .12, .12, .12, .18, .18)	.0013	74	6	6	2	2	2	0	0	0	0	0
(.13, .13, .13, .13, .13, .13, .13, .13)	.0000	70	8	8	0	0	0	0	0	0	0	0
(.01, .01, .01, .07, .09, .09, .09, .63)	.0377	> 96	7	7	4	4	3	3	3	3	3	3
(.03, .03, .03, .21, .07, .07, .07, .49)	.0221	> 96	6	6	6	6	3	3	3	3	0	0
(.01, .01, .03, .05, .09, .09, .27, .45)	.0213	> 96	6	6	4	4	4	4	3	3	2	2
(.02, .02, .04, .12, .08, .08, .16, .48)	.0201	> 96	5	5	5	3	3	3	3	2	2	0
(.01, .02, .02, .05, .09, .18, .18, .45)	.0192	> 96	5	5	4	4	4	4	3	3	3	1
(.05, .05, .05, .35, .05, .05, .05, .35)	.0169	> 96	6	6	6	6	6	6	0	0	0	0
(.02, .02, .08, .08, .08, .08, .32, .32)	.0133	> 96	6	6	6	2	2	2	2	2	2	0
(.02, .02, .02, .04, .18, .18, .18, .36)	.0131	> 96	4	4	4	4	4	4	4	3	3	0
(.04, .04, .08, .24, .06, .06, .12, .36)	.0117	> 96	5	5	5	4	4	2	2	0	0	0
(.03, .03, .09, .15, .07, .07, .21, .35)	.0105	> 96	5	5	4	4	2	2	2	2	0	0
(.02, .04, .06, .08, .08, .16, .24, .32)	.0099	> 96	5	5	5	3	3	2	2	1	1	0
(.03, .06, .06, .15, .07, .14, .14, .35)	.0090	> 96	4	4	4	4	3	1	1	1	0	0
(.05, .05, .15, .25, .05, .05, .15, .25)	.0069	> 96	4	4	4	4	4	4	0	0	0	0
(.04, .04, .16, .16, .06, .06, .24, .24)	.0065	> 96	4	4	4	4	4	2	2	0	0	0
(.04, .04, .06, .06, .16, .16, .24, .24)	.0065	> 96	4	4	4	4	4	2	2	0	0	0
(.06, .06, .06, .12, .14, .14, .14, .28)	.0047	> 96	3	3	3	3	3	0	0	0	0	0
(.04, .08, .12, .16, .06, .12, .18, .24)	.0039	> 96	3	3	3	2	2	1	1	0	0	0

TABLE: 3X3
 STATISTIC: KULLBACK
 ALPHA: .10

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.04,.06,.10,.06,.09,.15,.10,.15,.25)	.0037	90	6	4	3	3	3	1	1	0	0	0
(.04,.08,.08,.08,.16,.16,.08,.16,.16)	.0021	80	5	5	5	5	1	1	1	0	0	0
(.06,.06,.08,.09,.09,.12,.15,.15,.20)	.0020	80	6	5	5	3	2	2	0	0	0	0
(.09,.09,.12,.09,.09,.12,.12,.12,.16)	.0005	80	8	4	4	0	0	0	0	0	0	0
(.11,.11,.11,.11,.11,.11,.11,.11,.11)	.0000	63	9	9	9	0	0	0	0	0	0	0
(.01,.01,.08,.01,.01,.08,.08,.08,.64)	.0361	>108	8	8	4	4	4	4	4	4	4	0
(.01,.03,.06,.01,.03,.06,.08,.24,.48)	.0214	>108	7	7	6	6	4	4	4	2	2	0
(.01,.02,.07,.02,.04,.14,.07,.14,.49)	.0201	>108	6	6	6	4	4	4	3	3	1	0
(.02,.02,.06,.02,.02,.06,.16,.16,.48)	.0199	>108	6	6	6	6	4	4	4	4	0	0
(.02,.04,.04,.02,.04,.04,.16,.32,.32)	.0141	>108	6	6	6	6	6	6	2	2	0	0
(.01,.04,.05,.02,.08,.10,.07,.28,.35)	.0129	>108	6	6	5	4	4	3	2	2	1	0
(.01,.03,.06,.03,.09,.18,.06,.18,.36)	.0112	>108	6	5	5	5	3	3	3	1	1	0
(.02,.03,.05,.04,.06,.10,.14,.21,.35)	.0105	>108	5	5	5	5	4	3	2	1	0	0
(.02,.02,.06,.06,.06,.18,.12,.12,.36)	.0101	>108	5	5	5	5	2	2	2	2	0	0
(.04,.04,.12,.04,.04,.12,.12,.12,.36)	.0092	>108	4	4	4	4	4	4	0	0	0	0
(.03,.03,.04,.06,.06,.08,.21,.21,.28)	.0081	>108	6	6	5	5	3	3	2	0	0	0
(.01,.04,.05,.04,.16,.20,.05,.20,.25)	.0073	>108	5	5	5	5	5	3	1	1	1	0
(.02,.04,.04,.06,.12,.12,.12,.24,.24)	.0061	>108	4	4	4	4	3	3	1	1	0	0
(.02,.03,.05,.08,.12,.20,.10,.15,.25)	.0054	>108	4	4	3	3	3	2	2	1	0	0
(.04,.08,.08,.04,.08,.08,.12,.24,.24)	.0053	>108	6	6	2	2	2	2	0	0	0	0
(.03,.03,.04,.12,.12,.16,.15,.15,.20)	.0035	>108	3	3	3	3	3	3	2	0	0	0

TABLE: 2X5
 STATISTIC: KULLBACK
 ALPHA: .10

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.04,.08,.08,.08,.12,.06,.12,.12,.12,.18)	.0014	95	5	5	5	2	2	1	1	0	0	0
(.10,.10,.10,.10,.10,.10,.10,.10,.10,.10)	.0000	79	10	10	10	0	0	0	0	0	0	0
(.01,.01,.01,.01,.06,.09,.09,.09,.09,.54)	.0228	>100	9	5	5	5	4	4	4	4	4	0
(.03,.03,.03,.03,.18,.07,.07,.07,.07,.42)	.0132	>100	8	8	8	8	4	4	4	0	0	0
(.01,.01,.01,.03,.04,.09,.09,.09,.27,.36)	.0130	>100	8	5	5	5	5	5	4	3	3	0
(.02,.02,.02,.04,.10,.08,.08,.08,.16,.40)	.0118	>100	7	7	4	4	4	4	3	3	0	0
(.05,.05,.05,.05,.30,.05,.05,.05,.05,.30)	.0100	>100	8	8	8	8	8	0	0	0	0	0
(.01,.01,.02,.03,.03,.09,.09,.18,.27,.27)	.0097	>100	7	5	5	5	5	5	5	3	2	0
(.02,.02,.04,.04,.08,.08,.08,.16,.16,.32)	.0077	>100	7	7	4	4	4	4	2	2	0	0
(.04,.04,.04,.08,.20,.06,.06,.06,.12,.30)	.0066	>100	7	7	6	6	3	3	0	0	0	0
(.03,.03,.03,.09,.12,.07,.07,.07,.21,.28)	.0062	>100	7	6	6	6	3	3	3	0	0	0
(.02,.04,.04,.04,.06,.08,.16,.16,.16,.24)	.0050	>100	6	6	5	5	4	4	1	1	0	0
(.05,.05,.05,.15,.20,.05,.05,.05,.15,.20)	.0040	>100	6	6	6	6	6	0	0	0	0	0
(.03,.03,.06,.09,.09,.07,.07,.14,.21,.21)	.0039	>100	7	5	5	5	2	2	2	0	0	0
(.04,.04,.08,.08,.16,.06,.06,.12,.12,.24)	.0035	>100	6	6	4	4	2	2	0	0	0	0
(.05,.05,.10,.15,.15,.05,.05,.10,.15,.15)	.0020	>100	4	4	4	4	4	0	0	0	0	0

CRITICAL EXPECTED VALUE DISTRIBUTION

TABLE: 2X2
 STATISTIC: KULLBACK
 ALPHA: .05

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.04,.16,.16,.64)	.0531	96	1	1	1	1	1	1	1	0	0	0
(.04,.06,.36,.54)	.0441	96	2	2	2	2	2	1	1	0	0	0
(.05,.05,.45,.45)	.0400	96	2	2	2	2	2	2	0	0	0	0
(.06,.14,.24,.56)	.0361	75	1	1	1	1	1	1	0	0	0	0
(.08,.12,.32,.48)	.0259	54	2	2	2	2	1	1	0	0	0	0
(.10,.10,.40,.40)	.0225	48	2	2	2	2	2	2	0	0	0	0
(.09,.21,.21,.49)	.0216	46	3	1	1	1	1	1	0	0	0	0
(.12,.18,.28,.42)	.0129	38	2	2	2	2	1	1	0	0	0	0
(.15,.15,.35,.35)	.0100	36	2	2	2	2	2	0	0	0	0	0
(.16,.24,.24,.36)	.0051	36	3	3	1	1	1	0	0	0	0	0
(.20,.20,.30,.30)	.0025	37	2	2	2	0	0	0	0	0	0	0
(.25,.25,.25,.25)	.0000	37	4	0	0	0	0	0	0	0	0	0
(.01,.09,.09,.81)	.1056	> 96	3	3	1	1	1	1	1	1	1	1
(.02,.08,.18,.72)	.0769	> 96	2	2	2	1	1	1	1	1	1	0
(.03,.07,.27,.63)	.0564	> 96	2	2	2	2	1	1	1	1	0	0

CRITICAL EXPECTED VALUE DISTRIBUTION

TABLE: 2X3
 STATISTIC:KULLBACK
 ALPHA: .05

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.06,.06,.08,.24,.24,.32)	.0108	96	3	3	3	2	2	0	0	0	0	0
(.05,.15,.30,.05,.15,.30)	.0106	90	2	2	2	2	2	2	0	0	0	0
(.10,.10,.30,.10,.10,.30)	.0089	62	4	4	4	4	0	0	0	0	0	0
(.04,.16,.20,.06,.24,.30)	.0086	69	2	2	2	2	2	2	1	1	0	0
(.06,.12,.12,.14,.28,.28)	.0070	80	3	1	1	1	1	1	0	0	0	0
(.08,.12,.20,.12,.18,.30)	.0052	48	5	4	3	3	3	1	1	0	0	0
(.10,.20,.20,.10,.20,.20)	.0022	55	2	2	2	2	2	0	0	0	0	0
(.12,.12,.16,.18,.18,.24)	.0017	45	5	5	3	2	2	0	0	0	0	0
(.17,.17,.17,.17,.17,.17)	.0000	38	6	6	6	6	0	0	0	0	0	0
(.01,.01,.08,.09,.09,.72)	.0624	> 96	5	5	3	2	2	2	2	2	2	2
(.03,.03,.24,.07,.07,.56)	.0360	> 96	4	4	4	4	2	2	2	2	0	0
(.01,.03,.06,.09,.27,.54)	.0351	> 96	4	4	3	3	3	2	2	2	1	1
(.02,.04,.14,.08,.16,.56)	.0334	> 96	3	3	3	2	2	2	2	1	1	0
(.02,.02,.06,.18,.18,.54)	.0324	> 96	3	3	3	3	3	2	2	2	2	0
(.05,.05,.40,.05,.05,.40)	.0272	> 96	4	4	4	4	4	4	0	0	0	0
(.02,.04,.04,.18,.36,.36)	.0214	> 96	3	3	3	3	3	3	3	1	1	0
(.02,.08,.10,.08,.32,.40)	.0198	> 96	4	3	3	1	1	1	1	1	1	0
(.04,.08,.28,.06,.12,.42)	.0190	> 96	3	3	3	2	2	1	1	0	0	0
(.03,.09,.18,.07,.21,.42)	.0167	> 96	3	3	2	2	1	1	1	1	0	0
(.04,.06,.10,.16,.24,.40)	.0153	> 96	3	2	2	2	2	1	1	0	0	0
(.06,.06,.18,.14,.14,.42)	.0148	> 96	2	2	2	2	2	0	0	0	0	0

TABLE: 2X4
 STATISTIC: KULLBACK
 ALPHA: .05

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.05,.10,.10,.25,.05,.10,.10,.25)	.0056	96	6	2	2	2	2	2	0	0	0	0
(.06,.06,.06,.12,.14,.14,.14,.28)	.0047	96	3	3	3	3	3	0	0	0	0	0
(.10,.10,.10,.20,.10,.10,.10,.20)	.0019	79	6	6	6	0	0	0	0	0	0	0
(.08,.08,.12,.12,.12,.12,.18,.18)	.0013	82	6	2	2	2	0	0	0	0	0	0
(.13,.13,.13,.13,.13,.13,.13,.13)	.0000	80	0	0	0	0	0	0	0	0	0	0
(.01,.01,.01,.07,.09,.09,.09,.63)	.0377	> 96	7	7	4	4	3	3	3	3	3	3
(.03,.03,.03,.21,.07,.07,.07,.49)	.0221	> 96	6	6	6	6	3	3	3	3	0	0
(.01,.01,.03,.05,.09,.09,.27,.45)	.0213	> 96	6	6	4	4	4	4	3	3	2	2
(.01,.02,.02,.05,.09,.18,.18,.45)	.0192	> 96	5	5	4	4	4	4	3	3	3	1
(.02,.02,.04,.12,.08,.08,.16,.48)	.0201	> 96	5	5	5	3	3	3	3	2	2	0
(.05,.05,.05,.35,.05,.05,.05,.35)	.0169	> 96	6	6	6	6	6	6	0	0	0	0
(.02,.02,.08,.08,.08,.08,.32,.32)	.0133	> 96	6	6	6	2	2	2	2	2	2	0
(.02,.02,.02,.04,.18,.18,.18,.36)	.0131	> 96	4	4	4	4	4	4	4	3	3	0
(.04,.04,.08,.24,.06,.06,.12,.36)	.0117	> 96	5	5	5	4	4	2	2	0	0	0
(.03,.03,.09,.15,.07,.07,.21,.35)	.0105	> 96	5	5	4	4	2	2	2	2	0	0
(.02,.04,.06,.08,.08,.16,.24,.32)	.0099	> 96	5	5	5	3	3	2	2	1	1	0
(.03,.06,.06,.15,.07,.14,.14,.35)	.0090	> 96	4	4	4	4	3	1	1	1	0	0
(.05,.05,.15,.25,.05,.05,.15,.25)	.0069	> 96	4	4	4	4	4	4	0	0	0	0
(.04,.04,.16,.16,.06,.06,.24,.24)	.0065	> 96	4	4	4	4	4	2	2	0	0	0
(.04,.04,.06,.06,.16,.16,.24,.24)	.0065	> 96	4	4	4	4	4	2	2	0	0	0
(.04,.08,.12,.16,.06,.12,.18,.24)	.0039	> 96	3	3	3	2	2	1	1	0	0	0

TABLE: 3X3
 STATISTIC: KULLBACK
 ALPHA: .05

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.04,.06,.10,.06,.09,.15,.10,.15,.25)	.0037	72	6	6	6	4	3	3	1	1	0	0
(.04,.08,.08,.08,.16,.16,.08,.16,.16)	.0021	80	5	5	5	5	1	1	1	0	0	0
(.06,.06,.08,.09,.09,.12,.15,.15,.20)	.0020	74	6	6	5	5	3	2	0	0	0	0
(.09,.09,.12,.09,.09,.12,.12,.12,.16)	.0005	64	8	8	8	4	4	0	0	0	0	0
(.11,.11,.11,.11,.11,.11,.11,.11,.11)	.0000	51	9	9	9	9	9	0	0	0	0	0
(.01,.01,.08,.01,.01,.08,.08,.08,.64)	.0361	>108	8	8	4	4	4	4	4	4	4	0
(.01,.03,.06,.01,.03,.06,.08,.24,.48)	.0214	>108	7	7	6	6	4	4	4	2	2	0
(.01,.02,.07,.02,.04,.14,.07,.14,.49)	.0201	>108	6	6	6	4	4	4	3	3	1	0
(.02,.02,.06,.02,.02,.06,.16,.16,.48)	.0199	>108	6	6	6	6	4	4	4	4	0	0
(.02,.04,.04,.02,.04,.04,.16,.32,.32)	.0141	>108	6	6	6	6	6	6	2	2	0	0
(.01,.04,.05,.02,.08,.10,.07,.28,.35)	.0129	>108	6	6	5	4	4	3	2	2	1	0
(.01,.03,.06,.03,.09,.18,.06,.18,.36)	.0112	>108	6	5	5	5	3	3	3	1	1	0
(.02,.03,.05,.04,.06,.10,.14,.21,.35)	.0105	>108	5	5	5	5	4	3	2	1	0	0
(.02,.02,.06,.06,.06,.18,.12,.12,.36)	.0101	>108	5	5	5	5	2	2	2	2	0	0
(.04,.04,.12,.04,.04,.12,.12,.12,.36)	.0092	>108	4	4	4	4	4	4	0	0	0	0
(.03,.03,.04,.06,.06,.08,.21,.21,.28)	.0081	>108	6	6	5	5	3	3	2	0	0	0
(.01,.04,.05,.04,.16,.20,.05,.20,.25)	.0073	>108	5	5	5	5	5	3	1	1	1	0
(.02,.04,.04,.06,.12,.12,.12,.24,.24)	.0061	>108	4	4	4	4	3	3	1	1	0	0
(.02,.03,.05,.08,.12,.20,.10,.15,.25)	.0054	>108	4	4	3	3	3	2	2	1	0	0
(.04,.08,.08,.04,.08,.08,.12,.24,.24)	.0053	>108	6	6	2	2	2	2	0	0	0	0
(.03,.03,.04,.12,.12,.16,.15,.15,.20)	.0035	>108	3	3	3	3	3	3	2	0	0	0

TABLE: 2X5
 STATISTIC: KULLBACK
 ALPHA: .05

TICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.04,.08,.08,.08,.12,.06,.12,.12,.12,.18)	.0014	97	5	5	5	2	2	1	1	0	0	0
(.10,.10,.10,.10,.10,.10,.10,.10,.10,.10)	.0000	85	10	10	0	0	0	0	0	0	0	0
(.01,.01,.01,.01,.06,.09,.09,.09,.09,.54)	.0228	>100	9	5	5	5	4	4	4	4	4	0
(.03,.03,.03,.03,.18,.07,.07,.07,.07,.42)	.0132	>100	8	8	8	8	4	4	4	0	0	0
(.01,.01,.01,.03,.04,.09,.09,.09,.27,.36)	.0130	>100	8	5	5	5	5	5	4	3	3	0
(.02,.02,.02,.04,.10,.08,.08,.08,.16,.40)	.0118	>100	7	7	4	4	4	4	3	3	0	0
(.05,.05,.05,.05,.30,.05,.05,.05,.05,.30)	.0100	>100	8	8	8	8	8	0	0	0	0	0
(.01,.01,.02,.03,.03,.09,.09,.18,.27,.27)	.0097	>100	7	5	5	5	5	5	5	3	2	0
(.02,.02,.04,.04,.08,.08,.08,.16,.16,.32)	.0077	>100	7	7	4	4	4	4	2	2	0	0
(.04,.04,.04,.08,.20,.06,.06,.06,.12,.30)	.0066	>100	7	7	6	6	3	3	0	0	0	0
(.03,.03,.03,.09,.12,.07,.07,.07,.21,.28)	.0062	>100	7	6	6	6	3	3	3	0	0	0
(.02,.04,.04,.04,.06,.08,.16,.16,.16,.24)	.0050	>100	6	6	5	5	4	4	1	1	0	0
(.05,.05,.05,.15,.20,.05,.05,.05,.15,.20)	.0040	>100	6	6	6	6	6	0	0	0	0	0
(.03,.03,.06,.09,.09,.07,.07,.14,.21,.21)	.0039	>100	7	5	5	5	2	2	2	0	0	0
(.04,.04,.08,.08,.16,.06,.06,.12,.12,.24)	.0035	>100	6	6	4	4	2	2	0	0	0	0
(.05,.05,.10,.15,.15,.05,.05,.10,.15,.15)	.0020	>100	4	4	4	4	4	0	0	0	0	0

CRITICAL EXPECTED VALUE DISTRIBUTION

TABLE: 2X2
 STATISTIC: KULLBACK
 ALPHA: .01

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.04,.16,.16,.64)	.0531	84	1	1	1	1	1	1	1	0	0	0
(.06,.14,.24,.56)	.0361	70	2	1	1	1	1	1	0	0	0	0
(.08,.12,.32,.48)	.0259	52	2	2	2	2	1	1	0	0	0	0
(.10,.10,.40,.40)	.0225	48	2	2	2	2	2	2	0	0	0	0
(.09,.21,.21,.49)	.0216	40	3	3	1	1	1	1	1	0	0	0
(.12,.18,.28,.42)	.0129	30	3	3	2	2	2	1	1	0	0	0
(.15,.15,.35,.35)	.0100	30	2	2	2	2	2	2	0	0	0	0
(.16,.24,.24,.36)	.0051	22	4	4	4	3	3	1	1	0	0	0
(.20,.20,.30,.30)	.0025	21	4	4	4	4	2	2	0	0	0	0
(.25,.25,.25,.25)	.0000	20	4	4	4	4	4	0	0	0	0	0
(.01,.09,.09,.81)	.1056	> 96	3	3	1	1	1	1	1	1	1	1
(.02,.08,.18,.72)	.0769	> 96	2	2	2	1	1	1	1	1	1	0
(.03,.07,.27,.63)	.0564	> 96	2	2	2	2	1	1	1	1	0	0
(.04,.06,.36,.54)	.0441	> 96	2	2	2	2	2	1	1	0	0	0
(.05,.05,.45,.45)	.0400	> 96	2	2	2	2	2	2	0	0	0	0

TABLE: 2X3
 STATISTIC:KULLBACK
 ALPHA: .01

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.04,.08,.28,.06,.12,.42)	.0190	84	3	3	3	3	2	1	1	0	0	0
(.03,.09,.18,.07,.21,.42)	.0167	45	5	4	3	3	3	3	2	1	1	0
(.04,.06,.10,.16,.24,.40)	.0153	36	5	5	4	4	4	3	3	2	1	0
(.06,.06,.18,.14,.14,.42)	.0148	50	5	4	4	4	2	2	2	0	0	0
(.06,.06,.08,.24,.24,.32)	.0108	89	3	3	3	2	2	0	0	0	0	0
(.05,.15,.30,.05,.15,.30)	.0106	82	2	2	2	2	2	2	0	0	0	0
(.10,.10,.30,.10,.10,.30)	.0089	67	4	4	4	4	0	0	0	0	0	0
(.04,.16,.20,.06,.24,.30)	.0086	50	3	3	2	2	2	2	2	1	0	0
(.06,.12,.12,.14,.28,.28)	.0070	75	3	1	1	1	1	1	0	0	0	0
(.08,.12,.20,.12,.18,.30)	.0052	36	5	5	5	4	3	3	1	1	0	0
(.10,.20,.20,.10,.20,.20)	.0022	44	6	6	2	2	2	2	0	0	0	0
(.12,.12,.16,.18,.18,.24)	.0017	35	6	6	5	5	3	2	0	0	0	0
(.17,.17,.17,.17,.17,.17)	.0000	36	6	6	6	6	0	0	0	0	0	0
(.01,.01,.08,.09,.09,.72)	.0624	> 96	5	5	3	2	2	2	2	2	2	2
(.03,.03,.24,.07,.07,.56)	.0360	> 96	4	4	4	4	2	2	2	2	0	0
(.01,.03,.06,.09,.27,.54)	.0351	> 96	4	4	3	3	3	2	2	2	1	1
(.02,.04,.14,.08,.16,.56)	.0334	> 96	3	3	3	2	2	2	2	1	1	0
(.02,.02,.06,.18,.18,.54)	.0324	> 96	3	3	3	3	3	2	2	2	2	0
(.05,.05,.40,.05,.05,.40)	.0272	> 96	4	4	4	4	4	4	0	0	0	0
(.02,.04,.04,.18,.36,.36)	.0214	> 96	3	3	3	3	3	3	3	1	1	0
(.02,.08,.10,.08,.32,.40)	.0198	> 96	4	3	3	1	1	1	1	1	1	0

TABLE: 2X4
 STATISTIC:KULLBACK
 ALPHA: .01

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.04,.04,.08,.24,.06,.06,.12,.36)	.0117	78	6	5	5	5	4	4	2	0	0	0
(.03,.03,.09,.15,.07,.07,.21,.35)	.0105	50	6	6	6	5	5	5	4	2	2	0
(.03,.06,.06,.15,.07,.14,.14,.35)	.0090	40	7	7	7	7	6	4	4	4	1	0
(.05,.05,.15,.25,.05,.05,.15,.25)	.0069	85	4	4	4	4	4	4	0	0	0	0
(.05,.10,.10,.25,.05,.10,.10,.25)	.0056	82	6	6	2	2	2	2	0	0	0	0
(.06,.06,.06,.12,.14,.14,.14,.28)	.0047	82	4	3	3	3	3	3	0	0	0	0
(.04,.08,.12,.16,.06,.12,.18,.24)	.0039	72	5	5	3	3	3	2	1	1	0	0
(.10,.10,.10,.20,.10,.10,.10,.20)	.0019	70	6	6	6	0	0	0	0	0	0	0
(.08,.08,.12,.12,.12,.12,.18,.18)	.0013	80	6	2	2	2	0	0	0	0	0	0
(.13,.13,.13,.13,.13,.13,.13,.13)	.0000	70	8	8	0	0	0	0	0	0	0	0
(.01,.01,.01,.07,.09,.09,.09,.63)	.0377	> 96	7	7	4	4	3	3	3	3	3	3
(.03,.03,.03,.21,.07,.07,.07,.49)	.0221	> 96	6	6	6	6	3	3	3	3	0	0
(.01,.01,.03,.05,.09,.09,.27,.45)	.0213	> 96	6	6	4	4	4	4	3	3	2	2
(.02,.02,.04,.12,.08,.08,.16,.48)	.0201	> 96	5	5	5	3	3	3	3	2	2	0
(.01,.02,.02,.05,.09,.18,.18,.45)	.0192	> 96	5	5	4	4	4	4	3	3	3	1
(.05,.05,.05,.35,.05,.05,.05,.35)	.0169	> 96	6	6	6	6	6	6	0	0	0	0
(.02,.02,.08,.08,.08,.08,.32,.32)	.0133	> 96	6	6	6	2	2	2	2	2	2	0
(.02,.02,.02,.04,.18,.18,.18,.36)	.0131	> 96	4	4	4	4	4	4	4	3	3	0
(.02,.04,.06,.08,.08,.16,.24,.32)	.0099	> 96	5	5	5	3	3	2	2	1	1	0
(.04,.04,.06,.06,.16,.16,.24,.24)	.0065	> 96	4	4	4	4	4	2	2	0	0	0
(.04,.04,.16,.16,.06,.06,.24,.24)	.0065	> 96	4	4	4	4	4	2	2	0	0	0

TABLE: 3X3
 STATISTIC: KULLBACK
 ALPHA: .01

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.01,.04,.05,.04,.16,.20,.05,.20,.25)	.0073	90	5	5	5	5	5	5	3	1	1	1
(.02,.04,.04,.06,.12,.12,.12,.24,.24)	.0061	96	4	4	4	4	4	3	3	1	1	0
(.04,.06,.10,.06,.09,.15,.10,.15,.25)	.0037	74	6	6	6	4	3	3	1	1	0	0
(.03,.03,.04,.12,.12,.16,.15,.15,.20)	.0035	78	5	3	3	3	3	3	3	2	0	0
(.04,.08,.08,.08,.16,.16,.08,.16,.16)	.0021	72	5	5	5	5	5	1	1	1	0	0
(.06,.06,.08,.09,.09,.12,.15,.15,.20)	.0020	72	6	6	5	5	3	2	0	0	0	0
(.09,.09,.12,.09,.09,.12,.12,.12,.16)	.0005	67	8	8	4	4	0	0	0	0	0	0
(.11,.11,.11,.11,.11,.11,.11,.11,.11)	.0000	51	9	9	9	9	9	0	0	0	0	0
(.01,.01,.08,.01,.01,.08,.08,.08,.64)	.0361	>108	8	8	4	4	4	4	4	4	4	0
(.01,.03,.06,.01,.03,.06,.08,.24,.48)	.0214	>108	7	7	6	6	4	4	4	2	2	0
(.01,.02,.07,.02,.04,.14,.07,.14,.49)	.0201	>108	6	6	6	4	4	4	3	3	1	0
(.02,.02,.06,.02,.02,.06,.16,.16,.48)	.0199	>108	6	6	6	6	4	4	4	4	0	0
(.02,.04,.04,.02,.04,.04,.16,.32,.32)	.0141	>108	6	6	6	6	6	6	2	2	0	0
(.01,.04,.05,.02,.08,.10,.07,.28,.35)	.0129	>108	6	6	5	4	4	3	2	2	1	0
(.01,.03,.06,.03,.09,.18,.06,.18,.36)	.0112	>108	6	5	5	5	3	3	3	1	1	0
(.02,.03,.05,.04,.06,.10,.14,.21,.35)	.0105	>108	5	5	5	5	4	3	2	1	0	0
(.02,.02,.06,.06,.06,.18,.12,.12,.36)	.0101	>108	5	5	5	5	2	2	2	2	0	0
(.04,.04,.12,.04,.04,.12,.12,.12,.36)	.0092	>108	4	4	4	4	4	4	0	0	0	0
(.03,.03,.04,.06,.06,.08,.21,.21,.28)	.0081	>108	6	6	5	5	3	3	2	0	0	0
(.02,.03,.05,.08,.12,.20,.10,.15,.25)	.0054	>108	4	4	3	3	3	2	2	1	0	0
(.04,.08,.08,.04,.08,.08,.12,.24,.24)	.0053	>108	6	6	2	2	2	2	0	0	0	0

TABLE: 2X5
 STATISTIC: KULLBACK
 ALPHA: .01

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.05,.05,.05,.15,.20,.05,.05,.05,.15,.20)	.0040	96	6	6	6	6	6	6	0	0	0	0
(.04,.04,.08,.08,.16,.06,.06,.12,.12,.24)	.0035	94	6	6	6	4	4	2	2	0	0	0
(.05,.05,.10,.15,.15,.05,.05,.10,.15,.15)	.0020	86	6	6	4	4	4	4	0	0	0	0
(.04,.08,.08,.08,.12,.06,.12,.12,.12,.18)	.0014	85	5	5	5	5	2	1	1	0	0	0
(.10,.10,.10,.10,.10,.10,.10,.10,.10,.10)	.0000	60	10	10	10	10	0	0	0	0	0	0
(.01,.01,.01,.01,.06,.09,.09,.09,.09,.54)	.0228	>100	9	5	5	5	4	4	4	4	4	0
(.03,.03,.03,.03,.18,.07,.07,.07,.07,.42)	.0132	>100	8	8	8	8	4	4	4	0	0	0
(.01,.01,.01,.03,.04,.09,.09,.09,.27,.36)	.0130	>100	8	5	5	5	5	5	4	3	3	0
(.02,.02,.02,.04,.10,.08,.08,.08,.16,.40)	.0118	>100	7	7	4	4	4	4	3	3	0	0
(.05,.05,.05,.05,.30,.05,.05,.05,.05,.30)	.0100	>100	8	8	8	8	8	0	0	0	0	0
(.01,.01,.02,.03,.03,.09,.09,.18,.27,.27)	.0097	>100	7	5	5	5	5	5	5	3	2	0
(.02,.02,.04,.04,.08,.08,.08,.16,.16,.32)	.0077	>100	7	7	4	4	4	4	2	2	0	0
(.04,.04,.04,.08,.20,.06,.06,.06,.12,.30)	.0066	>100	7	7	6	6	3	3	0	0	0	0
(.03,.03,.03,.09,.12,.07,.07,.07,.21,.28)	.0062	>100	7	6	6	6	3	3	3	0	0	0
(.02,.04,.04,.04,.06,.08,.16,.16,.16,.24)	.0050	>100	6	6	5	5	4	4	1	1	0	0
(.03,.03,.06,.09,.09,.07,.07,.14,.21,.21)	.0039	>100	7	5	5	5	2	2	2	0	0	0

TABLE: S2X2X2
 STATISTIC: KULLBACK
 ALPHA: .10

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.048,.072,.072,.108,.112,.168,.168,.252)	.0040	71	5	5	5	3	3	1	1	0	0	0
(.075,.075,.075,.075,.175,.175,.175,.175)	.0025	99	4	4	4	0	0	0	0	0	0	0
(.064,.096,.096,.144,.096,.144,.144,.216)	.0020	74	4	4	4	1	1	1	0	0	0	0
(.100,.100,.100,.100,.150,.150,.150,.150)	.0006	54	8	8	4	4	4	0	0	0	0	0
(.125,.125,.125,.125,.125,.125,.125,.125)	.0000	58	8	8	8	0	0	0	0	0	0	0
(.008,.012,.032,.048,.072,.108,.288,.432)	.0206	>104	5	5	5	4	4	4	3	2	2	1
(.009,.021,.021,.049,.081,.189,.189,.441)	.0189	>104	5	5	4	4	4	3	3	3	1	1
(.016,.024,.064,.096,.064,.096,.256,.384)	.0144	>104	6	4	4	4	2	2	2	2	1	0
(.015,.015,.035,.035,.135,.135,.315,.315)	.0141	>104	4	4	4	4	4	4	4	2	2	0
(.018,.042,.042,.098,.072,.168,.168,.392)	.0130	>104	4	4	4	3	3	3	1	1	1	0
(.016,.024,.024,.036,.144,.216,.216,.324)	.0121	>104	4	4	4	4	4	4	4	3	1	0
(.025,.025,.025,.025,.225,.225,.225,.225)	.0100	>104	4	4	4	4	4	4	4	4	0	0
(.030,.030,.070,.070,.120,.120,.280,.280)	.0090	>104	4	4	4	2	2	2	2	2	0	0
(.027,.063,.063,.147,.063,.147,.147,.343)	.0088	>104	4	4	4	4	1	1	1	1	1	0
(.032,.048,.048,.072,.128,.192,.192,.288)	.0074	>104	4	4	4	3	3	3	1	0	0	0
(.050,.050,.050,.050,.200,.200,.200,.200)	.0056	>104	4	4	4	4	4	0	0	0	0	0
(.045,.045,.105,.105,.105,.105,.245,.245)	.0054	>104	2	2	2	2	2	2	0	0	0	0

TABLE: 2X2X2
 STATISTIC:KULLBACK
 ALPHA: .10

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.050,.050,.050,.050,.200,.200,.200,.200)	.0056	96	4	4	4	4	4	4	0	0	0	0
(.045,.045,.105,.105,.105,.105,.245,.245)	.0054	96	2	2	2	2	2	2	0	0	0	0
(.075,.075,.075,.075,.175,.175,.175,.175)	.0025	63	4	4	4	4	4	4	0	0	0	0
(.064,.096,.096,.144,.096,.144,.144,.216)	.0020	66	7	4	4	4	1	1	0	0	0	0
(.100,.100,.100,.100,.150,.150,.150,.150)	.0006	63	8	4	4	4	0	0	0	0	0	0
(.125,.125,.125,.125,.125,.125,.125,.125)	.0000	56	8	8	8	0	0	0	0	0	0	0
(.001,.009,.009,.081,.009,.081,.081,.729)	.0533	> 96	7	7	7	4	4	4	4	4	4	4
(.003,.007,.027,.063,.027,.063,.243,.567)	.0331	> 96	6	6	6	6	4	4	4	4	2	2
(.004,.016,.016,.064,.036,.144,.144,.576)	.0318	> 96	5	5	5	5	4	4	4	3	3	1
(.005,.005,.045,.045,.045,.045,.405,.405)	.0264	> 96	6	6	6	6	6	6	2	2	2	2
(.008,.032,.032,.128,.032,.128,.128,.512)	.0237	> 96	4	4	4	4	4	4	4	1	1	1
(.008,.012,.032,.048,.072,.108,.288,.432)	.0206	> 96	5	5	5	5	4	4	3	2	2	1
(.009,.021,.021,.049,.081,.189,.189,.441)	.0189	> 96	5	5	5	4	4	4	3	3	1	1
(.016,.024,.064,.096,.064,.096,.256,.384)	.0144	> 96	6	4	4	4	2	2	2	2	1	0
(.015,.015,.035,.035,.135,.135,.315,.315)	.0141	> 96	4	4	4	4	4	4	4	2	2	0
(.018,.042,.042,.098,.072,.168,.168,.392)	.0130	> 96	5	4	4	4	3	3	1	1	1	0
(.016,.024,.024,.036,.144,.216,.216,.324)	.0121	> 96	4	4	4	4	4	4	4	3	1	0
(.025,.025,.025,.025,.225,.225,.225,.225)	.0100	> 96	4	4	4	4	4	4	4	4	0	0
(.030,.030,.070,.070,.120,.120,.280,.280)	.0090	> 96	4	4	4	4	2	2	2	2	0	0
(.027,.063,.063,.147,.063,.147,.147,.343)	.0088	> 96	4	4	4	4	1	1	1	1	0	0
(.032,.048,.048,.072,.128,.192,.192,.288)	.0074	> 96	4	4	4	4	3	3	1	0	0	0
(.048,.072,.072,.108,.112,.168,.168,.252)	.0040	> 96	3	3	3	3	1	1	0	0	0	0

TABLE: 2X2X3
 STATISTIC: KULLBACK
 ALPHA: .10

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.083,.083,.083,.083,.083,.083,.083,.083,.083,.083,.083)	.0000	96	12	12	0	0	0	0	0	0	0	0
(.001,.001,.008,.009,.009,.072,.009,.009,.072,.081,.081,.648)	.0300	> 96	11	11	11	9	7	7	7	7	7	7
(.002,.004,.014,.008,.016,.056,.018,.036,.126,.072,.144,.504)	.0181	> 96	9	9	9	9	8	7	7	6	6	3
(.002,.002,.006,.018,.018,.054,.018,.018,.054,.162,.162,.486)	.0177	> 96	9	9	9	9	9	7	7	7	7	3
(.005,.005,.040,.005,.005,.040,.045,.045,.360,.045,.045,.360)	.0156	> 96	10	10	10	10	10	10	6	4	4	4
(.004,.008,.028,.016,.032,.112,.016,.032,.112,.064,.128,.448)	.0139	> 96	8	8	8	8	7	7	7	5	4	2
(.010,.010,.080,.010,.010,.080,.040,.040,.320,.040,.040,.320)	.0118	> 96	10	10	10	8	8	8	8	4	4	4
(.003,.009,.018,.007,.021,.042,.027,.081,.162,.063,.189,.378)	.0113	> 96	9	9	9	8	7	7	6	6	4	3
(.004,.006,.010,.016,.024,.040,.036,.054,.090,.144,.216,.360)	.0107	> 96	9	9	8	8	8	7	7	5	4	3
(.009,.018,.063,.021,.042,.147,.021,.042,.147,.049,.098,.343)	.0082	> 96	9	8	8	8	7	7	4	4	2	1
(.006,.018,.036,.014,.042,.084,.024,.072,.144,.056,.168,.336)	.0082	> 96	9	9	8	8	7	6	5	4	3	1
(.010,.010,.030,.010,.010,.030,.090,.090,.270,.090,.090,.270)	.0081	> 96	10	10	6	6	6	6	6	6	4	4
(.004,.016,.020,.006,.024,.030,.036,.144,.180,.054,.216,.270)	.0080	> 96	8	8	8	8	8	7	7	6	4	2
(.016,.016,.128,.024,.024,.192,.024,.024,.192,.036,.036,.288)	.0079	> 96	8	8	8	8	8	8	8	6	2	0
(.008,.012,.020,.032,.048,.080,.032,.048,.080,.128,.192,.320)	.0077	> 96	9	9	9	7	7	7	5	3	3	1
(.006,.012,.012,.014,.028,.028,.054,.108,.108,.126,.252,.252)	.0073	> 96	7	7	7	7	7	6	6	6	4	1
(.020,.020,.060,.020,.020,.060,.080,.080,.240,.080,.080,.240)	.0055	> 96	10	10	10	6	6	4	4	4	4	0
(.008,.032,.040,.012,.048,.060,.032,.128,.160,.048,.192,.240)	.0054	> 96	8	8	8	8	7	5	2	2	2	1
(.012,.012,.016,.018,.018,.024,.108,.108,.144,.162,.162,.216)	.0051	> 96	6	6	6	6	6	6	6	6	5	0
(.012,.024,.024,.028,.056,.056,.048,.096,.096,.112,.224,.224)	.0049	> 96	9	7	7	7	7	5	4	4	1	0
(.020,.040,.140,.020,.040,.140,.030,.060,.210,.030,.060,.210)	.0048	> 96	8	8	8	8	8	6	6	4	2	0
(.012,.036,.072,.018,.054,.108,.028,.084,.168,.042,.126,.252)	.0046	> 96	8	8	7	7	6	5	4	3	2	0
(.018,.027,.045,.042,.063,.105,.042,.063,.105,.098,.147,.245)	.0037	> 96	8	7	7	7	5	5	2	2	1	0
(.015,.060,.075,.015,.060,.075,.035,.140,.175,.035,.140,.175)	.0032	> 96	8	8	8	6	6	4	4	2	2	0
(.024,.024,.032,.036,.036,.048,.096,.096,.128,.144,.144,.192)	.0031	> 96	8	6	6	6	6	6	5	2	0	0
(.032,.032,.096,.048,.048,.144,.048,.048,.144,.072,.072,.216)	.0030	> 96	9	8	8	8	6	6	2	0	0	0
(.025,.075,.150,.025,.075,.150,.025,.075,.150,.025,.075,.150)	.0026	> 96	8	8	8	4	4	4	4	4	0	0
(.024,.048,.048,.036,.072,.072,.056,.112,.112,.084,.168,.168)	.0021	> 96	8	8	7	7	5	4	2	1	0	0
(.040,.060,.100,.040,.060,.100,.060,.090,.150,.060,.090,.150)	.0013	> 96	10	8	6	6	6	2	2	0	0	0
(.045,.045,.060,.045,.045,.060,.105,.105,.140,.105,.105,.140)	.0013	> 96	6	6	6	6	6	4	0	0	0	0
(.050,.100,.100,.050,.100,.100,.050,.100,.100,.050,.100,.100)	.0006	> 96	12	4	4	4	4	4	0	0	0	0

TABLE: 2X2X2
 STATISTIC: KULLBACK
 ALPHA: .05

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.027,.063,.063,.147,.063,.147,.147,.343)	.0088	72	4	4	4	4	4	4	1	1	1	0
(.075,.075,.075,.075,.175,.175,.175,.175)	.0025	63	4	4	4	4	4	4	0	0	0	0
(.064,.096,.096,.144,.096,.144,.144,.216)	.0020	49	7	7	7	4	4	4	1	0	0	0
(.100,.100,.100,.100,.150,.150,.150,.150)	.0006	65	8	4	4	4	0	0	0	0	0	0
(.125,.125,.125,.125,.125,.125,.125,.125)	.0000	48	8	8	8	8	0	0	0	0	0	0
(.001,.009,.009,.081,.009,.081,.081,.729)	.0533	> 96	7	7	7	4	4	4	4	4	4	4
(.003,.007,.027,.063,.027,.063,.243,.567)	.0331	> 96	6	6	6	6	4	4	4	4	2	2
(.004,.016,.016,.064,.036,.144,.144,.576)	.0318	> 96	5	5	5	5	4	4	4	3	3	1
(.005,.005,.045,.045,.045,.045,.405,.405)	.0264	> 96	6	6	6	6	6	6	2	2	2	2
(.008,.032,.032,.128,.032,.128,.128,.512)	.0237	> 96	4	4	4	4	4	4	4	1	1	1
(.008,.012,.032,.048,.072,.108,.288,.432)	.0206	> 96	5	5	5	5	4	4	3	2	2	1
(.009,.021,.021,.049,.081,.189,.189,.441)	.0189	> 96	5	5	5	4	4	4	3	3	1	1
(.016,.024,.064,.096,.064,.096,.256,.384)	.0144	> 96	6	4	4	4	2	2	2	2	1	0
(.015,.015,.035,.035,.135,.135,.315,.315)	.0141	> 96	4	4	4	4	4	4	4	2	2	0
(.018,.042,.042,.098,.072,.168,.168,.392)	.0130	> 96	5	4	4	4	3	3	1	1	1	0
(.016,.024,.024,.036,.144,.216,.216,.324)	.0121	> 96	4	4	4	4	4	4	4	3	1	0
(.025,.025,.025,.025,.225,.225,.225,.225)	.0100	> 96	4	4	4	4	4	4	4	4	0	0
(.030,.030,.070,.070,.120,.120,.280,.280)	.0090	> 96	4	4	4	4	2	2	2	2	0	0
(.032,.048,.048,.072,.128,.192,.192,.288)	.0074	> 96	4	4	4	4	3	3	1	0	0	0
(.050,.050,.050,.050,.200,.200,.200,.200)	.0056	> 96	4	4	4	4	4	4	0	0	0	0
(.045,.045,.105,.105,.105,.105,.245,.245)	.0054	> 96	2	2	2	2	2	2	0	0	0	0
(.048,.072,.072,.108,.112,.168,.168,.252)	.0040	> 96	3	3	3	3	1	1	0	0	0	0

TABLE: 2X2X3
 STATISTIC: KULLBACK
 ALPHA: .05

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.083,.083,.083,.083,.083,.083,.083,.083,.083,.083,.083,.083)	.0000	96	12	12	0	0	0	0	0	0	0	0
(.001,.001,.008,.009,.009,.072,.009,.009,.072,.081,.081,.648)	.0300	> 96	11	11	11	9	7	7	7	7	7	7
(.002,.004,.014,.008,.016,.056,.018,.036,.126,.072,.144,.504)	.0181	> 96	9	9	9	9	8	7	7	6	6	3
(.002,.002,.006,.018,.018,.054,.018,.018,.054,.162,.162,.486)	.0177	> 96	9	9	9	9	9	7	7	7	7	3
(.005,.005,.040,.005,.005,.040,.045,.045,.360,.045,.045,.360)	.0156	> 96	10	10	10	10	10	6	4	4	4	4
(.004,.008,.028,.016,.032,.112,.016,.032,.112,.064,.128,.448)	.0139	> 96	8	8	8	8	7	7	7	5	4	2
(.010,.010,.080,.010,.010,.080,.040,.040,.320,.040,.040,.320)	.0118	> 96	10	10	10	8	8	8	8	4	4	4
(.003,.009,.018,.007,.021,.042,.027,.081,.162,.063,.189,.378)	.0113	> 96	9	9	9	8	7	7	6	6	4	3
(.004,.006,.010,.016,.024,.040,.036,.054,.090,.144,.216,.360)	.0107	> 96	9	9	8	8	8	7	7	5	4	3
(.009,.018,.063,.021,.042,.147,.021,.042,.147,.049,.098,.343)	.0082	> 96	9	8	8	8	7	7	4	4	2	1
(.006,.018,.036,.014,.042,.084,.024,.072,.144,.056,.168,.336)	.0082	> 96	9	9	8	8	7	6	5	4	3	1
(.010,.010,.030,.010,.010,.030,.090,.090,.270,.090,.090,.270)	.0081	> 96	10	10	6	6	6	6	6	6	4	4
(.004,.016,.020,.006,.024,.030,.036,.144,.180,.054,.216,.270)	.0080	> 96	8	8	8	8	8	7	7	6	4	2
(.016,.016,.128,.024,.024,.192,.024,.024,.192,.036,.036,.288)	.0079	> 96	8	8	8	8	8	8	8	6	2	0
(.008,.012,.020,.032,.048,.080,.032,.048,.080,.128,.192,.320)	.0077	> 96	9	9	9	7	7	7	5	3	3	1
(.006,.012,.012,.014,.028,.028,.054,.108,.108,.126,.252,.252)	.0073	> 96	7	7	7	7	7	6	6	6	4	1
(.020,.020,.060,.020,.020,.060,.080,.080,.240,.080,.080,.240)	.0055	> 96	10	10	10	6	6	4	4	4	4	0
(.008,.032,.040,.012,.048,.060,.032,.128,.160,.048,.192,.240)	.0054	> 96	8	8	8	8	8	7	5	2	2	1
(.012,.012,.016,.018,.018,.024,.108,.108,.144,.162,.162,.216)	.0051	> 96	6	6	6	6	6	6	6	6	5	0
(.012,.024,.024,.028,.056,.056,.048,.096,.096,.112,.224,.224)	.0049	> 96	9	7	7	7	7	5	4	4	1	0
(.020,.040,.140,.020,.040,.140,.030,.060,.210,.030,.060,.210)	.0048	> 96	8	8	8	8	8	6	6	4	2	0
(.012,.036,.072,.018,.054,.108,.028,.084,.166,.042,.126,.252)	.0046	> 96	8	8	7	7	6	5	4	3	2	0
(.018,.027,.045,.042,.063,.105,.042,.063,.105,.098,.147,.245)	.0037	> 96	8	7	7	7	5	5	2	2	1	0
(.015,.060,.075,.015,.060,.075,.035,.140,.175,.035,.140,.175)	.0032	> 96	8	8	8	6	6	4	4	2	2	0
(.024,.024,.032,.036,.036,.048,.096,.096,.128,.144,.144,.192)	.0031	> 96	8	6	6	6	6	6	5	2	0	0
(.032,.032,.096,.048,.048,.144,.048,.048,.144,.072,.072,.216)	.0030	> 96	9	8	8	8	6	6	2	0	0	0
(.025,.075,.150,.025,.075,.150,.025,.075,.150,.025,.075,.150)	.0026	> 96	8	8	8	4	4	4	4	4	0	0
(.024,.048,.048,.036,.072,.072,.056,.112,.112,.084,.168,.168)	.0021	> 96	8	8	7	7	5	4	2	1	0	0
(.040,.060,.100,.040,.060,.100,.060,.090,.150,.060,.090,.150)	.0013	> 96	10	8	6	6	6	2	2	0	0	0
(.045,.045,.060,.045,.045,.060,.105,.105,.140,.105,.105,.140)	.0013	> 96	6	6	6	6	6	4	0	0	0	0
(.050,.100,.100,.050,.100,.100,.050,.100,.100,.050,.100,.100)	.0006	> 96	12	4	4	4	4	4	0	0	0	0

TABLE: S2X2X2
 STATISTIC: KULLBACK
 ALPHA: .01

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.048,.072,.072,.108,.112,.168,.168,.252)	.0040	74	5	5	4	3	3	1	1	0	0	0
(.075,.075,.075,.075,.175,.175,.175,.175)	.0025	81	4	4	4	4	0	0	0	0	0	0
(.064,.096,.096,.144,.096,.144,.144,.216)	.0020	64	7	4	4	4	1	1	0	0	0	0
(.100,.100,.100,.100,.150,.150,.150,.150)	.0006	59	8	8	4	4	4	0	0	0	0	0
(.125,.125,.125,.125,.125,.125,.125,.125)	.0000	51	8	8	8	8	0	0	0	0	0	0
(.008,.012,.032,.048,.072,.108,.288,.432)	.0206	>104	5	5	5	4	4	4	3	2	2	1
(.009,.021,.021,.049,.081,.189,.189,.441)	.0189	>104	5	5	4	4	4	3	3	3	1	1
(.016,.024,.064,.096,.064,.096,.256,.384)	.0144	>104	6	4	4	4	2	2	2	2	1	0
(.015,.015,.035,.035,.135,.135,.315,.315)	.0141	>104	4	4	4	4	4	4	4	2	2	0
(.018,.042,.042,.098,.072,.168,.168,.392)	.0130	>104	4	4	4	3	3	3	1	1	1	0
(.016,.024,.024,.036,.144,.216,.216,.324)	.0121	>104	4	4	4	4	4	4	4	3	1	0
(.025,.025,.025,.025,.225,.225,.225,.225)	.0100	>104	4	4	4	4	4	4	4	4	0	0
(.030,.030,.070,.070,.120,.120,.280,.280)	.0090	>104	4	4	4	2	2	2	2	0	0	0
(.027,.063,.063,.147,.063,.147,.147,.343)	.0088	>104	4	4	4	4	1	1	1	1	0	0
(.032,.048,.048,.072,.128,.192,.192,.288)	.0074	>104	4	4	4	3	3	3	1	0	0	0
(.050,.050,.050,.050,.200,.200,.200,.200)	.0056	>104	4	4	4	4	4	0	0	0	0	0
(.045,.045,.105,.105,.105,.105,.245,.245)	.0054	>104	2	2	2	2	2	2	0	0	0	0

TABLE: 2X2X2
 STATISTIC: KULLBACK
 ALPHA: .01

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.030,.030,.070,.070,.120,.120,.280,.280)	.0090	83	6	4	4	4	4	2	2	2	0	0
(.027,.063,.063,.147,.063,.147,.147,.343)	.0088	56	7	7	4	4	4	4	4	1	1	0
(.050,.050,.050,.050,.200,.200,.200,.200)	.0056	68	4	4	4	4	4	4	4	0	0	0
(.048,.072,.072,.108,.112,.168,.168,.252)	.0040	64	5	5	5	4	3	3	1	0	0	0
(.075,.075,.075,.075,.175,.175,.175,.175)	.0025	61	4	4	4	4	4	4	0	0	0	0
(.064,.096,.096,.144,.096,.144,.144,.216)	.0020	53	7	7	7	4	4	1	1	0	0	0
(.100,.100,.100,.100,.150,.150,.150,.150)	.0006	45	8	8	8	8	4	4	0	0	0	0
(.125,.125,.125,.125,.125,.125,.125,.125)	.0000	45	8	8	8	8	8	0	0	0	0	0
(.001,.009,.009,.081,.009,.081,.081,.729)	.0533	> 96	7	7	7	4	4	4	4	4	4	4
(.003,.007,.027,.063,.027,.063,.243,.567)	.0331	> 96	6	6	6	6	4	4	4	4	2	2
(.004,.016,.016,.064,.036,.144,.144,.576)	.0318	> 96	5	5	5	5	4	4	4	3	3	1
(.005,.005,.045,.045,.045,.045,.405,.405)	.0264	> 96	6	6	6	6	6	6	2	2	2	2
(.008,.032,.032,.128,.032,.128,.128,.512)	.0237	> 96	4	4	4	4	4	4	4	1	1	1
(.008,.012,.032,.048,.072,.108,.288,.432)	.0206	> 96	5	5	5	5	4	4	3	2	2	1
(.009,.021,.021,.049,.081,.189,.189,.441)	.0189	> 96	5	5	5	4	4	4	3	3	1	1
(.016,.024,.064,.096,.064,.096,.256,.384)	.0144	> 96	6	4	4	4	2	2	2	2	1	0
(.015,.015,.035,.035,.135,.135,.315,.315)	.0141	> 96	4	4	4	4	4	4	4	2	2	0
(.018,.042,.042,.098,.072,.168,.168,.392)	.0130	> 96	5	4	4	4	3	3	1	1	1	0
(.016,.024,.024,.036,.144,.216,.216,.324)	.0121	> 96	4	4	4	4	4	4	4	3	1	0
(.025,.025,.025,.025,.225,.225,.225,.225)	.0100	> 96	4	4	4	4	4	4	4	4	0	0
(.032,.048,.048,.072,.128,.192,.192,.288)	.0074	> 96	4	4	4	4	3	3	1	0	0	0
(.045,.045,.105,.105,.105,.105,.245,.245)	.0054	> 96	2	2	2	2	2	2	0	0	0	0

CRITICAL EXPECTED VALUE DISTRIBUTION

TABLE: 2X2
 STATISTIC: PEARSON
 ALPHA: .10

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.03,.07,.27,.63)	.0564	46	2	2	2	2	2	2	2	1	1	0
(.04,.16,.16,.64)	.0531	36	3	3	3	3	3	1	1	1	1	0
(.04,.06,.36,.54)	.0441	29	2	2	2	2	2	2	2	2	2	0
(.05,.05,.45,.45)	.0400	27	2	2	2	2	2	2	2	2	2	0
(.06,.14,.24,.56)	.0361	< 8	4	4	4	4	4	4	3	3	3	1
(.08,.12,.32,.48)	.0259	< 8	4	4	4	4	4	4	4	3	2	2
(.10,.10,.40,.40)	.0225	9	4	4	4	4	4	4	4	2	2	2
(.09,.21,.21,.49)	.0216	13	4	4	4	4	3	3	3	3	1	0
(.12,.18,.28,.42)	.0129	21	4	4	3	3	3	2	2	1	0	0
(.15,.15,.35,.35)	.0100	22	4	4	4	2	2	2	2	0	0	0
(.16,.24,.24,.36)	.0051	22	4	4	4	3	3	1	1	0	0	0
(.20,.20,.30,.30)	.0025	23	4	4	4	4	2	2	0	0	0	0
(.25,.25,.25,.25)	.0000	23	4	4	4	4	4	0	0	0	0	0
(.01,.09,.09,.81)	.1056	> 96	3	3	1	1	1	1	1	1	1	1
(.02,.08,.18,.72)	.0769	> 96	2	2	2	1	1	1	1	1	1	0

TABLE: 2X3
 STATISTIC: PEARSON
 ALPHA: .10

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.03,.03,.24,.07,.07,.56)	.0360	30	5	5	5	4	4	4	4	4	2	2
(.01,.03,.06,.09,.27,.54)	.0351	17	6	5	5	5	5	5	4	4	4	2
(.02,.04,.14,.08,.16,.56)	.0334	14	6	6	6	5	5	5	5	5	4	2
(.05,.05,.40,.05,.05,.40)	.0272	32	4	4	4	4	4	4	4	4	4	0
(.02,.04,.04,.18,.36,.36)	.0214	36	4	4	4	4	3	3	3	3	3	1
(.02,.08,.10,.08,.32,.40)	.0198	<12	6	6	6	6	6	6	5	4	4	3
(.04,.08,.28,.06,.12,.42)	.0190	30	5	5	4	4	4	4	4	3	2	0
(.03,.09,.18,.07,.21,.42)	.0167	<12	6	6	6	6	6	5	5	5	3	2
(.04,.06,.10,.16,.24,.40)	.0153	<12	6	6	6	6	6	6	5	5	4	2
(.06,.06,.18,.14,.14,.42)	.0148	22	6	5	5	5	5	5	5	2	2	0
(.06,.06,.08,.24,.24,.32)	.0108	<12	6	6	6	6	6	6	6	5	3	3
(.05,.15,.30,.05,.15,.30)	.0106	24	6	6	6	4	4	4	4	2	2	0
(.10,.10,.30,.10,.10,.30)	.0089	<12	6	6	6	6	6	6	6	4	4	0
(.04,.16,.20,.06,.24,.30)	.0086	<12	6	6	6	6	6	6	6	5	3	2
(.06,.12,.12,.14,.28,.28)	.0070	<12	6	6	6	6	6	6	6	4	4	1
(.08,.12,.20,.12,.18,.30)	.0052	<12	6	6	6	6	6	6	6	5	3	1
(.10,.20,.20,.10,.20,.20)	.0022	<12	6	6	6	6	6	6	6	6	2	0
(.12,.12,.16,.18,.18,.24)	.0017	<12	6	6	6	6	6	6	6	6	3	0
(.17,.17,.17,.17,.17,.17)	.0000	<12	6	6	6	6	6	6	6	6	0	0
(.01,.01,.08,.09,.09,.72)	.0624	> 96	5	5	3	2	2	2	2	2	2	2
(.02,.02,.06,.18,.18,.54)	.0324	> 96	3	3	3	3	3	2	2	2	2	0

CRITICAL EXPECTED VALUE DISTRIBUTION

TABLE: 2X2
 STATISTIC: PEAKSON
 ALPHA: .05

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.03,.07,.27,.63)	.0564	52	2	2	2	2	2	2	2	1	1	0
(.04,.16,.16,.64)	.0531	38	3	3	3	3	1	1	1	1	1	0
(.04,.06,.36,.54)	.0441	48	2	2	2	2	2	2	2	2	1	0
(.05,.05,.45,.45)	.0400	44	2	2	2	2	2	2	2	2	0	0
(.06,.14,.24,.56)	.0361	16	4	4	3	3	3	3	3	2	1	1
(.08,.12,.32,.48)	.0259	18	4	4	3	3	3	2	2	2	1	0
(.10,.10,.40,.40)	.0225	17	4	4	4	4	2	2	2	2	2	0
(.09,.21,.21,.49)	.0216	< 8	4	4	4	4	4	4	4	3	3	1
(.12,.18,.28,.42)	.0129	< 8	4	4	4	4	4	4	4	3	3	1
(.15,.15,.35,.35)	.0100	< 8	4	4	4	4	4	4	4	4	2	0
(.16,.24,.24,.36)	.0051	17	4	4	4	4	3	3	1	1	0	0
(.20,.20,.30,.30)	.0025	18	4	4	4	4	4	2	2	0	0	0
(.25,.25,.25,.25)	.0000	18	4	4	4	4	4	4	0	0	0	0
(.01,.09,.09,.81)	.1056	> 96	3	3	1	1	1	1	1	1	1	1
(.02,.08,.18,.72)	.0769	> 96	2	2	2	1	1	1	1	1	1	0

CRITICAL EXPECTED VALUE DISTRIBUTION

TABLE: 2X2
 STATISTIC: PEARSON
 ALPHA: .01

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.02,.08,.18,.72)	.0769	46	3	3	2	2	2	2	2	1	1	1
(.03,.07,.27,.63)	.0564	10	4	4	4	4	3	3	3	3	2	2
(.04,.16,.16,.64)	.0531	21	3	3	3	3	3	3	3	1	1	1
(.04,.06,.36,.54)	.0441	48	2	2	2	2	2	2	2	2	1	0
(.05,.05,.45,.45)	.0400	56	2	2	2	2	2	2	2	2	0	0
(.06,.14,.24,.56)	.0361	9	4	4	4	4	4	3	3	3	2	1
(.08,.12,.32,.48)	.0259	< 8	4	4	4	4	4	4	4	3	2	2
(.10,.10,.40,.40)	.0225	16	4	4	4	4	2	2	2	2	2	0
(.09,.21,.21,.49)	.0216	< 8	4	4	4	4	4	4	4	3	3	1
(.12,.18,.28,.42)	.0129	< 8	4	4	4	4	4	4	4	3	2	1
(.15,.15,.35,.35)	.0100	< 8	4	4	4	4	4	4	4	4	2	0
(.16,.24,.24,.36)	.0051	< 8	4	4	4	4	4	4	4	4	3	0
(.20,.20,.30,.30)	.0025	< 8	4	4	4	4	4	4	4	4	2	0
(.25,.25,.25,.25)	.0000	< 8	4	4	4	4	4	4	4	4	0	0
(.01,.09,.09,.81)	.1056	> 96	3	3	1	1	1	1	1	1	1	1

CRITICAL EXPECTED VALUE DISTRIBUTION

TABLE: 2X3
 STATISTIC: PEARSON
 ALPHA: .01

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.01,.03,.06,.09,.27,.54)	.0351	81	4	4	4	3	3	3	2	2	1	1
(.02,.04,.14,.08,.16,.56)	.0334	30	5	5	5	5	5	5	3	3	2	1
(.02,.02,.06,.18,.18,.54)	.0324	24	5	5	5	5	5	5	3	3	3	2
(.05,.05,.40,.05,.05,.40)	.0272	61	4	4	4	4	4	4	4	0	0	0
(.02,.04,.04,.18,.36,.36)	.0214	13	6	6	6	6	6	6	4	4	3	3
(.02,.08,.10,.08,.32,.40)	.0198	<12	6	6	6	6	6	6	5	4	4	3
(.04,.08,.28,.06,.12,.42)	.0190	30	5	5	4	4	4	4	4	3	2	0
(.03,.09,.18,.07,.21,.42)	.0167	<12	6	6	6	6	6	5	5	5	3	2
(.04,.06,.10,.16,.24,.40)	.0153	<12	6	6	6	6	6	6	5	5	4	2
(.06,.06,.18,.14,.14,.42)	.0148	<12	6	6	6	6	6	5	5	5	4	2
(.06,.06,.08,.24,.24,.32)	.0108	13	6	6	6	6	6	6	5	3	3	2
(.05,.15,.30,.05,.15,.30)	.0106	32	6	4	4	4	4	4	2	2	2	0
(.10,.10,.30,.10,.10,.30)	.0089	24	6	6	6	4	4	4	4	4	0	0
(.04,.16,.20,.06,.24,.30)	.0086	21	6	6	6	6	5	4	3	2	2	1
(.06,.12,.12,.14,.28,.28)	.0070	<12	6	6	6	6	6	6	6	4	4	1
(.08,.12,.20,.12,.18,.30)	.0052	20	6	6	6	6	5	5	4	3	1	0
(.10,.20,.20,.10,.20,.20)	.0022	22	6	6	6	6	6	6	2	2	0	0
(.12,.12,.16,.18,.18,.24)	.0017	15	6	6	6	6	6	6	6	5	2	0
(.17,.17,.17,.17,.17,.17)	.0000	15	6	6	6	6	6	6	6	6	0	0
(.01,.01,.08,.09,.09,.72)	.0624	> 96	5	5	3	2	2	2	2	2	2	2
(.03,.03,.24,.07,.07,.56)	.0360	> 96	4	4	4	4	2	2	2	0	0	0

TABLE: 2X4
 STATISTIC: PEARSON
 ALPHA: .01

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.01,.01,.03,.05,.09,.09,.27,.45)	.0213	94	6	6	4	4	4	4	3	3	2	2
(.02,.02,.04,.12,.08,.08,.16,.48)	.0201	25	7	7	7	7	7	6	5	3	2	
(.01,.02,.02,.05,.09,.18,.18,.45)	.0192	80	5	5	5	4	4	4	3	3	3	1
(.05,.05,.05,.35,.05,.05,.05,.35)	.0169	68	6	6	6	6	6	6	6	0	0	0
(.02,.02,.08,.08,.08,.08,.32,.32)	.0133	32	6	6	6	6	6	6	6	6	2	2
(.02,.02,.02,.04,.18,.18,.18,.36)	.0131	<16	8	8	8	8	8	7	7	7	4	4
(.04,.04,.08,.24,.06,.06,.12,.36)	.0117	80	6	5	5	5	4	4	2	0	0	0
(.03,.03,.09,.15,.07,.07,.21,.35)	.0105	20	8	8	8	7	7	7	6	5	5	2
(.02,.04,.06,.08,.08,.16,.24,.32)	.0099	<16	8	8	8	8	8	7	7	6	5	3
(.03,.06,.06,.15,.07,.14,.14,.35)	.0090	<16	8	8	8	8	8	7	7	7	4	3
(.05,.05,.15,.25,.05,.05,.15,.25)	.0069	55	6	6	4	4	4	4	4	4	0	0
(.04,.04,.16,.16,.06,.06,.24,.24)	.0065	55	6	6	4	4	4	4	4	2	0	0
(.04,.04,.06,.06,.16,.16,.24,.24)	.0065	<16	8	8	8	8	8	8	8	6	4	4
(.05,.10,.10,.25,.05,.10,.10,.25)	.0056	37	8	6	6	6	6	6	6	2	2	0
(.06,.06,.06,.12,.14,.14,.14,.28)	.0047	<16	8	8	8	8	8	8	7	7	4	3
(.04,.08,.12,.16,.06,.12,.18,.24)	.0039	40	8	7	7	6	5	5	3	2	1	0
(.10,.10,.10,.20,.10,.10,.10,.20)	.0019	23	8	8	8	8	8	8	6	6	0	0
(.08,.08,.12,.12,.12,.12,.18,.18)	.0013	30	8	8	8	8	8	6	6	2	0	0
(.13,.13,.13,.13,.13,.13,.13,.13)	.0000	24	8	8	8	8	8	8	8	0	0	0
(.01,.01,.01,.07,.09,.09,.09,.63)	.0377	> 96	7	7	4	4	3	3	3	3	3	3
(.03,.03,.03,.21,.07,.07,.07,.49)	.0221	> 96	6	6	6	6	3	3	3	3	0	0

TABLE: 3X3
 STATISTIC: PEARSON
 ALPHA: .01

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.01,.03,.06,.01,.03,.06,.08,.24,.48)	.0214	99	7	7	7	6	6	4	4	4	2	2
(.01,.02,.07,.02,.04,.14,.07,.14,.49)	.0201	91	6	6	6	6	4	4	4	3	3	1
(.02,.02,.06,.02,.02,.06,.16,.16,.48)	.0199	79	6	6	6	6	6	6	4	4	4	0
(.02,.04,.04,.02,.04,.04,.16,.32,.32)	.0141	<18	9	9	9	9	9	7	7	7	6	6
(.01,.04,.05,.02,.08,.10,.07,.28,.35)	.0129	52	7	7	7	7	7	6	5	4	2	1
(.01,.03,.06,.03,.09,.18,.06,.18,.36)	.0112	46	8	8	6	6	6	6	5	5	3	1
(.02,.03,.05,.04,.06,.10,.14,.21,.35)	.0105	18	9	9	9	9	8	8	8	7	6	4
(.02,.02,.06,.06,.06,.18,.12,.12,.36)	.0101	<18	9	9	9	9	8	8	8	7	5	2
(.04,.04,.12,.04,.04,.12,.12,.12,.36)	.0092	<18	9	9	9	9	8	8	8	8	4	4
(.03,.03,.04,.06,.06,.08,.21,.21,.28)	.0081	30	9	9	8	8	6	6	6	6	5	2
(.01,.04,.05,.04,.16,.20,.05,.20,.25)	.0073	27	9	9	9	9	8	6	5	5	5	1
(.02,.04,.04,.06,.12,.12,.12,.24,.24)	.0061	20	9	9	9	9	9	9	7	7	4	3
(.02,.03,.05,.08,.12,.20,.10,.15,.25)	.0054	<18	9	9	9	9	9	9	8	7	5	3
(.04,.08,.08,.04,.08,.08,.12,.24,.24)	.0053	20	9	9	9	9	9	9	7	7	6	2
(.04,.06,.10,.06,.09,.15,.10,.15,.25)	.0037	<18	9	9	9	9	9	9	8	8	6	1
(.03,.03,.04,.12,.12,.16,.15,.15,.20)	.0035	<18	9	9	9	9	9	9	9	8	3	3
(.04,.08,.08,.08,.16,.16,.08,.16,.16)	.0021	25	9	9	9	9	9	9	5	5	1	0
(.06,.06,.08,.09,.09,.12,.15,.15,.20)	.0020	<18	9	9	9	9	9	9	9	8	5	0
(.09,.09,.12,.09,.09,.12,.12,.12,.16)	.0005	22	9	9	9	9	9	9	9	8	4	0
(.11,.11,.11,.11,.11,.11,.11,.11,.11)	.0000	<18	9	9	9	9	9	9	9	9	0	0
(.01,.01,.08,.01,.01,.08,.08,.08,.64)	.0361	>108	8	8	4	4	4	4	4	4	4	0
(.01,.01,.08,.01,.01,.08,.08,.08,.64)	.0361	>108	8	8	4	4	4	4	4	4	4	0

TABLE: 2X5
 STATISTIC: PEARSON
 ALPHA: .01

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.03,.03,.03,.03,.18,.07,.07,.07,.07,.42)	.0132	46	9	9	8	8	8	8	8	4	4	0
(.01,.01,.01,.03,.04,.09,.09,.09,.27,.36)	.0130	85	8	8	8	5	5	5	5	4	3	3
(.02,.02,.02,.04,.10,.08,.08,.08,.16,.40)	.0118	55	9	9	8	8	8	7	4	4	3	0
(.05,.05,.05,.05,.30,.05,.05,.05,.05,.30)	.0100	58	8	8	8	8	8	8	8	8	0	0
(.01,.01,.02,.03,.03,.09,.09,.18,.27,.27)	.0097	54	8	7	7	7	7	7	5	5	5	2
(.02,.02,.04,.04,.08,.08,.08,.16,.16,.32)	.0077	<20	10	10	10	10	9	9	9	7	7	4
(.04,.04,.04,.08,.20,.06,.06,.06,.12,.30)	.0066	44	9	9	8	8	8	7	7	6	3	0
(.03,.03,.03,.09,.12,.07,.07,.07,.21,.28)	.0062	56	8	8	8	8	7	6	6	3	3	0
(.02,.04,.04,.04,.06,.08,.16,.16,.16,.24)	.0050	60	9	6	6	6	6	6	5	4	1	0
(.05,.05,.05,.15,.20,.05,.05,.05,.15,.20)	.0040	49	10	8	8	6	6	6	6	6	0	0
(.03,.03,.06,.09,.09,.07,.07,.14,.21,.21)	.0039	33	10	10	10	10	8	8	7	7	3	2
(.04,.04,.08,.08,.16,.06,.06,.12,.12,.24)	.0035	54	9	9	8	8	6	6	4	2	0	0
(.05,.05,.10,.15,.15,.05,.05,.10,.15,.15)	.0020	32	10	10	10	10	10	10	6	4	4	0
(.04,.08,.08,.08,.12,.06,.12,.12,.12,.18)	.0014	28	10	10	10	10	10	9	9	5	2	0
(.10,.10,.10,.10,.10,.10,.10,.10,.10,.10)	.0000	26	10	10	10	10	10	10	10	10	0	0
(.01,.01,.01,.01,.06,.09,.09,.09,.09,.54)	.0228	>100	9	5	5	5	4	4	4	4	4	0
(.01,.01,.01,.01,.06,.09,.09,.09,.09,.54)	.0228	>100	9	5	5	5	4	4	4	4	4	0

TABLE: S2X2X2
 STATISTIC: PEARSON
 ALPHA: .10

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.050,.050,.050,.050,.200,.200,.200,.200)	.0056	93	4	4	4	4	4	4	0	0	0	0
(.048,.072,.072,.108,.112,.168,.168,.252)	.0040	56	7	5	5	5	3	3	1	1	0	0
(.075,.075,.075,.075,.175,.175,.175,.175)	.0025	44	8	8	8	4	4	4	4	0	0	0
(.064,.096,.096,.144,.096,.144,.144,.216)	.0020	56	7	7	4	4	4	1	1	0	0	0
(.100,.100,.100,.100,.150,.150,.150,.150)	.0006	47	8	8	8	4	4	4	0	0	0	0
(.125,.125,.125,.125,.125,.125,.125,.125)	.0000	38	8	8	8	8	8	8	0	0	0	0
(.008,.012,.032,.048,.072,.108,.288,.432)	.0206	>104	5	5	5	4	4	4	3	2	2	1
(.009,.021,.021,.049,.081,.189,.189,.441)	.0189	>104	5	5	4	4	4	3	3	3	1	1
(.016,.024,.064,.096,.064,.096,.256,.384)	.0144	>104	6	4	4	4	2	2	2	2	1	0
(.015,.015,.035,.035,.135,.135,.315,.315)	.0141	>104	4	4	4	4	4	4	4	2	2	0
(.018,.042,.042,.098,.072,.168,.168,.392)	.0130	>104	4	4	4	3	3	3	1	1	1	0
(.016,.024,.024,.036,.144,.216,.216,.324)	.0121	>104	4	4	4	4	4	4	4	3	1	0
(.025,.025,.025,.025,.225,.225,.225,.225)	.0100	>104	4	4	4	4	4	4	4	4	0	0
(.030,.030,.070,.070,.120,.120,.280,.280)	.0090	>104	4	4	4	2	2	2	2	0	0	0
(.027,.063,.063,.147,.063,.147,.147,.343)	.0088	>104	4	4	4	4	1	1	1	1	0	0
(.032,.048,.048,.072,.128,.192,.192,.288)	.0074	>104	4	4	4	3	3	3	1	0	0	0
(.045,.045,.105,.105,.105,.105,.245,.245)	.0054	>104	2	2	2	2	2	2	0	0	0	0

TABLE: 2X2X2
 STATISTIC: PEARSON
 ALPHA: .10

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.003,.007,.027,.063,.027,.063,.243,.567)	.0331	63	6	6	6	6	6	6	6	4	4	2
(.004,.016,.016,.064,.036,.144,.144,.576)	.0318	37	7	7	7	7	7	5	5	5	4	3
(.005,.005,.045,.045,.045,.045,.405,.405)	.0264	56	6	6	6	6	6	6	6	6	2	2
(.008,.032,.032,.128,.032,.128,.128,.512)	.0237	19	8	7	7	7	7	7	7	7	4	4
(.008,.012,.032,.048,.072,.108,.288,.432)	.0206	27	7	7	7	6	6	6	6	6	5	3
(.009,.021,.021,.049,.081,.189,.189,.441)	.0189	<16	8	8	8	7	7	7	7	5	5	4
(.016,.024,.064,.096,.064,.096,.256,.384)	.0144	<16	8	8	8	8	7	7	6	6	6	2
(.015,.015,.035,.035,.135,.135,.315,.315)	.0141	<16	8	8	8	8	8	6	6	6	4	4
(.018,.042,.042,.098,.072,.168,.168,.392)	.0130	<16	8	8	8	8	7	7	7	7	5	3
(.016,.024,.024,.036,.144,.216,.216,.324)	.0121	37	7	7	7	5	5	4	4	4	4	3
(.025,.025,.025,.025,.225,.225,.225,.225)	.0100	43	8	4	4	4	4	4	4	4	4	0
(.030,.030,.070,.070,.120,.120,.280,.280)	.0090	<16	8	8	8	8	8	8	6	6	6	2
(.027,.063,.063,.147,.063,.147,.147,.343)	.0088	<16	8	8	8	8	8	7	7	7	4	1
(.032,.048,.048,.072,.128,.192,.192,.288)	.0074	<16	8	8	8	8	8	8	7	5	4	3
(.050,.050,.050,.050,.200,.200,.200,.200)	.0056	<16	8	8	8	8	8	8	8	4	4	4
(.045,.045,.105,.105,.105,.105,.245,.245)	.0054	<16	8	8	8	8	8	8	8	6	6	2
(.048,.072,.072,.108,.112,.168,.168,.252)	.0040	<16	8	8	8	8	8	8	7	7	5	1
(.075,.075,.075,.075,.175,.175,.175,.175)	.0025	<16	8	8	8	8	8	8	8	8	4	0
(.064,.096,.096,.144,.096,.144,.144,.216)	.0020	<16	8	8	8	8	8	8	8	7	4	0
(.100,.100,.100,.100,.150,.150,.150,.150)	.0006	<16	8	8	8	8	8	8	8	8	4	0
(.125,.125,.125,.125,.125,.125,.125,.125)	.0000	<16	8	8	8	8	8	8	8	8	0	0
(.001,.009,.009,.081,.009,.081,.081,.729)	.0533	> 96	7	7	7	4	4	4	4	4	4	4

TABLE: 2X2X3
STATISTIC: PEARSON
ALPHA: .10

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.002,.002,.006,.018,.018,.054,.018,.018,.054,.162,.162,.486)	.0177	96	9	9	9	9	9	7	7	7	7	3
(.005,.005,.040,.005,.005,.040,.045,.045,.360,.045,.045,.360)	.0156	96	10	10	10	10	10	6	4	4	4	4
(.004,.008,.028,.016,.032,.112,.016,.032,.112,.064,.128,.448)	.0139	43	11	11	11	11	11	8	8	7	4	
(.010,.010,.080,.010,.010,.080,.040,.040,.320,.040,.040,.320)	.0118	<24	12	12	12	10	10	10	10	10	8	
(.003,.009,.018,.007,.021,.042,.027,.081,.162,.063,.189,.378)	.0113	50	11	10	9	9	9	9	8	7	6	4
(.004,.006,.010,.016,.024,.040,.036,.054,.090,.144,.216,.360)	.0107	42	11	10	10	10	9	9	9	8	7	4
(.009,.018,.063,.021,.042,.147,.021,.042,.147,.049,.098,.343)	.0082	<24	12	12	11	11	11	11	9	8	4	
(.006,.018,.036,.014,.042,.084,.024,.072,.144,.056,.168,.336)	.0082	<24	12	12	11	11	11	11	10	9	8	5
(.010,.010,.030,.010,.010,.030,.090,.090,.270,.090,.090,.270)	.0081	<24	12	12	12	12	10	10	10	10	6	6
(.004,.016,.020,.006,.024,.030,.036,.144,.180,.054,.216,.270)	.0080	<24	12	12	12	12	11	10	9	8	8	7
(.016,.016,.128,.024,.024,.192,.024,.024,.192,.036,.036,.288)	.0079	32	12	11	11	11	9	9	8	8	8	6
(.008,.012,.020,.032,.048,.080,.032,.048,.080,.128,.192,.320)	.0077	25	12	12	11	11	11	11	10	9	7	5
(.006,.012,.012,.014,.028,.028,.054,.108,.108,.126,.252,.252)	.0073	24	12	12	12	12	10	10	10	9	7	6
(.020,.020,.060,.020,.020,.060,.080,.080,.240,.080,.080,.240)	.0055	<24	12	12	12	12	12	10	10	10	10	4
(.008,.032,.040,.012,.048,.060,.032,.128,.160,.048,.192,.240)	.0054	27	12	12	12	12	11	10	9	8	8	4
(.012,.012,.016,.018,.018,.024,.108,.108,.144,.162,.162,.216)	.0051	<24	12	12	12	12	12	11	11	8	6	6
(.012,.024,.024,.028,.036,.056,.048,.096,.096,.112,.224,.224)	.0049	<24	12	12	12	12	12	10	10	10	7	4
(.020,.040,.140,.020,.040,.140,.030,.060,.210,.030,.060,.210)	.0048	44	12	10	10	10	8	8	8	8	6	2
(.012,.036,.072,.018,.054,.108,.028,.084,.168,.042,.126,.252)	.0046	<24	12	12	12	12	11	11	10	9	7	4
(.018,.027,.045,.042,.063,.105,.042,.063,.105,.098,.147,.245)	.0037	<24	12	12	12	12	12	11	11	10	7	2
(.015,.060,.075,.015,.060,.075,.035,.140,.175,.035,.140,.175)	.0032	<24	12	12	12	12	12	12	10	8	8	4
(.024,.024,.032,.036,.036,.048,.096,.096,.128,.144,.144,.192)	.0031	<24	12	12	12	12	12	12	11	8	6	5
(.032,.032,.096,.048,.048,.144,.048,.048,.144,.072,.072,.216)	.0030	<24	12	12	12	12	12	12	11	11	9	8
(.025,.075,.150,.025,.075,.150,.025,.075,.150,.025,.075,.150)	.0026	<24	12	12	12	12	12	12	12	8	8	4
(.024,.048,.048,.036,.072,.072,.056,.112,.112,.084,.168,.168)	.0021	<24	12	12	12	12	12	12	10	10	7	2
(.040,.060,.100,.040,.060,.100,.060,.090,.150,.060,.090,.150)	.0013	<24	12	12	12	12	12	12	12	10	6	2
(.045,.045,.060,.045,.045,.060,.105,.105,.140,.105,.105,.140)	.0013	<24	12	12	12	12	12	12	12	10	6	0
(.050,.100,.100,.050,.100,.100,.050,.100,.100,.050,.100,.100)	.0006	<24	12	12	12	12	12	12	12	12	4	0
(.083,.083,.083,.083,.083,.083,.083,.083,.083,.083,.083,.083)	.0000	<24	12	12	12	12	12	12	12	12	0	0
(.001,.001,.008,.009,.009,.072,.009,.009,.072,.081,.081,.648)	.0300	> 96	11	11	11	9	7	7	7	7	7	7
(.002,.004,.014,.008,.016,.056,.018,.036,.126,.072,.144,.504)	.0181	> 96	9	9	9	9	8	7	7	6	6	3

TABLE: 2X2X2
 STATISTIC: PEARSON
 ALPHA: .05

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.004,.016,.016,.064,.036,.144,.144,.576)	.0318	88	5	5	5	5	5	4	4	3	3	1
(.005,.005,.045,.045,.045,.045,.405,.405)	.0264	73	6	6	6	6	6	6	6	2	2	2
(.008,.032,.032,.128,.032,.128,.128,.512)	.0237	35	7	7	7	7	7	7	4	4	4	1
(.008,.012,.032,.048,.072,.108,.288,.432)	.0206	31	7	7	6	6	6	6	6	5	4	3
(.009,.021,.021,.049,.081,.189,.189,.441)	.0189	23	7	7	7	7	7	7	5	5	5	3
(.016,.024,.064,.096,.064,.096,.256,.384)	.0144	<16	8	8	8	8	7	7	6	6	6	2
(.015,.015,.035,.035,.135,.135,.315,.315)	.0141	<16	8	8	8	8	8	6	6	6	4	4
(.018,.042,.042,.098,.072,.168,.168,.392)	.0130	<16	8	8	8	8	7	7	7	7	5	3
(.016,.024,.024,.036,.144,.216,.216,.324)	.0121	34	7	7	7	5	5	5	4	4	4	3
(.025,.025,.025,.025,.225,.225,.225,.225)	.0100	56	4	4	4	4	4	4	4	4	4	0
(.030,.030,.070,.070,.120,.120,.280,.280)	.0090	<16	8	8	8	8	8	8	6	6	6	2
(.027,.063,.063,.147,.063,.147,.147,.343)	.0088	<16	8	8	8	8	8	7	7	7	4	1
(.032,.048,.048,.072,.128,.192,.192,.288)	.0074	<16	8	8	8	8	8	8	7	5	4	3
(.050,.050,.050,.050,.200,.200,.200,.200)	.0056	<16	8	8	8	8	8	8	8	4	4	4
(.045,.045,.105,.105,.105,.105,.245,.245)	.0054	<16	8	8	8	8	8	8	8	6	6	2
(.048,.072,.072,.108,.112,.168,.168,.252)	.0040	<16	8	8	8	8	8	8	7	7	5	1
(.075,.075,.075,.075,.175,.175,.175,.175)	.0025	<16	8	8	8	8	8	8	8	8	4	0
(.064,.096,.096,.144,.096,.144,.144,.216)	.0020	<16	8	8	8	8	8	8	8	7	4	0
(.100,.100,.100,.100,.150,.150,.150,.150)	.0006	<16	8	8	8	8	8	8	8	8	4	0
(.125,.125,.125,.125,.125,.125,.125,.125)	.0000	<16	8	8	8	8	8	8	8	8	0	0
(.001,.009,.009,.081,.009,.081,.081,.729)	.0533	> 96	7	7	7	4	4	4	4	4	4	4
(.003,.007,.027,.063,.027,.063,.243,.567)	.0331	> 96	6	6	6	6	4	4	4	4	2	2

TABLE: 2X2X3
STATISTIC: PEARSON
ALPHA: .05

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.004,.008,.028,.016,.032,.112,.016,.032,.112,.064,.128,.448)	.0139	76	1110	8	8	8	8	7	7	4	2	
(.010,.010,.080,.010,.010,.080,.040,.040,.320,.040,.040,.320)	.0118	31	1210	10	10	10	10	10	10	8	4	
(.003,.009,.018,.007,.021,.042,.027,.081,.162,.063,.189,.378)	.0113	63	9	9	9	9	8	8	7	6	3	
(.004,.006,.010,.016,.024,.040,.036,.054,.090,.144,.216,.360)	.0107	71	9	9	9	9	8	8	8	7	5	3
(.009,.018,.063,.021,.042,.147,.021,.042,.147,.049,.098,.343)	.0082	<24	1212	11	11	11	11	11	11	9	8	4
(.006,.018,.036,.014,.042,.084,.024,.072,.144,.056,.168,.336)	.0082	36	1111	11	11	10	9	9	8	6	4	
(.010,.010,.030,.010,.010,.030,.090,.090,.270,.090,.090,.270)	.0081	26	1212	12	10	10	10	10	10	6	6	
(.004,.016,.020,.006,.024,.030,.036,.144,.180,.054,.216,.270)	.0080	49	1010	9	8	8	8	8	8	7	4	
(.016,.016,.128,.024,.024,.192,.024,.024,.192,.036,.036,.288)	.0079	55	9	9	9	8	8	8	8	8	2	
(.008,.012,.020,.032,.048,.080,.032,.048,.080,.128,.192,.320)	.0077	26	1212	11	11	11	11	10	9	7	5	
(.006,.012,.012,.014,.028,.028,.054,.108,.108,.126,.252,.252)	.0073	25	1212	12	12	10	10	10	9	7	6	
(.020,.020,.060,.020,.020,.060,.080,.080,.240,.080,.080,.240)	.0055	<24	1212	12	12	12	10	10	10	10	4	
(.008,.032,.040,.012,.048,.060,.032,.128,.160,.048,.192,.240)	.0054	<24	1212	12	12	12	11	11	10	8	8	5
(.012,.012,.016,.018,.018,.024,.108,.108,.144,.162,.162,.216)	.0051	<24	1212	12	12	12	11	11	11	8	6	6
(.012,.024,.024,.028,.056,.056,.048,.096,.096,.112,.224,.224)	.0049	<24	1212	12	12	12	12	10	10	10	7	4
(.020,.040,.140,.020,.040,.140,.030,.060,.210,.030,.060,.210)	.0048	41	1212	10	10	10	8	8	8	6	2	
(.012,.036,.072,.018,.054,.108,.028,.084,.168,.042,.126,.252)	.0046	34	1212	11	11	11	10	9	8	6	3	
(.018,.027,.045,.042,.063,.105,.042,.063,.105,.098,.147,.245)	.0037	<24	1212	12	12	12	12	11	11	10	7	2
(.015,.060,.075,.015,.060,.075,.035,.140,.175,.035,.140,.175)	.0032	<24	1212	12	12	12	12	12	10	8	8	4
(.024,.024,.032,.036,.036,.048,.096,.096,.128,.144,.144,.192)	.0031	46	1212	11	11	9	8	6	6	5	0	
(.032,.032,.096,.048,.048,.144,.048,.048,.144,.072,.072,.216)	.0030	<24	1212	12	12	12	12	11	11	9	8	2
(.025,.075,.150,.025,.075,.150,.025,.075,.150,.025,.075,.150)	.0026	<24	1212	12	12	12	12	12	8	8	4	
(.024,.048,.048,.036,.072,.072,.056,.112,.112,.084,.168,.168)	.0021	29	1212	12	12	12	12	12	10	8	5	1
(.040,.060,.100,.040,.060,.100,.060,.090,.150,.060,.090,.150)	.0013	<24	1212	12	12	12	12	12	12	10	6	2
(.045,.045,.060,.045,.045,.060,.105,.105,.140,.105,.105,.140)	.0013	<24	1212	12	12	12	12	12	12	10	6	0
(.050,.100,.100,.050,.100,.100,.050,.100,.100,.050,.100,.100)	.0006	29	1212	12	12	12	12	12	12	12	4	0
(.083,.083,.083,.083,.083,.083,.083,.083,.083,.083,.083,.083)	.0000	<24	1212	12	12	12	12	12	12	12	0	0
(.001,.001,.008,.009,.009,.072,.009,.009,.072,.081,.081,.648)	.0300	> 96	1111	11	9	7	7	7	7	7	7	7
(.002,.004,.014,.008,.016,.056,.018,.036,.126,.072,.144,.504)	.0181	> 96	9	9	9	9	8	7	7	6	6	3
(.002,.002,.006,.018,.018,.054,.018,.018,.054,.162,.162,.486)	.0177	> 96	9	9	9	9	9	7	7	7	7	3
(.005,.005,.040,.005,.005,.040,.045,.045,.360,.045,.045,.360)	.0156	> 96	1010	10	10	10	10	10	6	4	4	4

TABLE: S2X2X2
 STATISTIC: PEARSON
 ALPHA: .01

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.075,.075,.075,.075,.175,.175,.175,.175)	.0025	72	4	4	4	4	4	0	0	0	0	0
(.064,.096,.096,.144,.096,.144,.144,.216)	.0020	56	7	7	4	4	4	1	1	0	0	0
(.100,.100,.100,.100,.150,.150,.150,.150)	.0006	40	8	8	8	8	4	4	0	0	0	0
(.125,.125,.125,.125,.125,.125,.125,.125)	.0000	37	8	8	8	8	8	8	0	0	0	0
(.008,.012,.032,.048,.072,.108,.288,.432)	.0206	>104	5	5	5	4	4	4	3	2	2	1
(.009,.021,.021,.049,.081,.189,.189,.441)	.0189	>104	5	5	4	4	4	3	3	3	1	1
(.016,.021,.064,.096,.064,.096,.256,.384)	.0144	>104	6	4	4	4	2	2	2	2	1	0
(.015,.015,.035,.035,.135,.135,.315,.315)	.0141	>104	4	4	4	4	4	4	4	2	2	0
(.018,.042,.042,.098,.072,.168,.168,.392)	.0130	>104	4	4	4	3	3	3	1	1	1	0
(.016,.024,.024,.036,.144,.216,.216,.324)	.0121	>104	4	4	4	4	4	4	4	3	1	0
(.025,.025,.025,.025,.225,.225,.225,.225)	.0100	>104	4	4	4	4	4	4	4	4	0	0
(.030,.030,.070,.070,.120,.120,.280,.280)	.0090	>104	4	4	4	2	2	2	2	0	0	0
(.027,.063,.063,.147,.063,.147,.147,.343)	.0088	>104	4	4	4	4	1	1	1	1	0	0
(.032,.048,.048,.072,.128,.192,.192,.288)	.0074	>104	4	4	4	3	3	3	1	0	0	0
(.050,.050,.050,.050,.200,.200,.200,.200)	.0056	>104	4	4	4	4	4	0	0	0	0	0
(.045,.045,.105,.105,.105,.105,.245,.245)	.0054	>104	2	2	2	2	2	2	0	0	0	0
(.048,.072,.072,.108,.112,.168,.168,.252)	.0040	>104	3	3	3	1	1	1	0	0	0	0

TABLE: 2X2X3
STATISTIC: PEARSON
ALPHA: .01

CRITICAL EXPECTED VALUE DISTRIBUTION

P VECTOR	MU2	N(MIN)	NUMBER OF EXPECTED VALUES LESS THAN									
			10	9	8	7	6	5	4	3	2	1
(.010,.010,.080,.010,.010,.080,.040,.040,.320,.040,.040,.320)	.0118	92	101010	8	8	8	8	4	4	4		
(.009,.018,.063,.021,.042,.147,.021,.042,.147,.049,.098,.343)	.0082	36	1111111111	9	9	8	7	4				
(.006,.018,.036,.014,.042,.084,.024,.072,.144,.056,.168,.336)	.0082	78	9	9	9	9	8	7	6	5	4	1
(.010,.010,.030,.010,.010,.030,.090,.090,.270,.090,.090,.270)	.0081	42	10101010101010	6	6	4						
(.016,.016,.128,.024,.024,.192,.024,.024,.192,.036,.036,.288)	.0079	76	9	8	8	8	8	8	8	6	0	
(.008,.012,.020,.032,.048,.080,.032,.048,.080,.128,.192,.320)	.0077	84	9	9	9	9	7	7	5	5	3	1
(.006,.012,.012,.014,.028,.028,.054,.108,.108,.126,.252,.252)	.0073	63	101010	9	7	7	7	6	6	4		
(.020,.020,.060,.020,.020,.060,.080,.080,.240,.080,.080,.240)	.0055	<24	1212121212101010	4								
(.008,.032,.040,.012,.048,.060,.032,.128,.160,.048,.192,.240)	.0054	76	9	8	8	8	8	7	4	2	2	
(.012,.012,.016,.018,.018,.024,.108,.108,.144,.162,.162,.216)	.0051	34	1212121111	9	8	6	6	6				
(.012,.024,.024,.028,.056,.056,.096,.096,.112,.224,.224)	.0049	<24	1212121212101010	7	4							
(.020,.040,.140,.020,.040,.140,.030,.060,.210,.030,.060,.210)	.0048	25	12121212121010	8	8	4						
(.012,.036,.072,.018,.054,.108,.028,.084,.168,.042,.126,.252)	.0046	<24	12121212111110	9	7	4						
(.018,.027,.045,.042,.063,.105,.042,.063,.105,.098,.147,.245)	.0037	<24	121212121211110	7	2							
(.015,.060,.075,.015,.060,.075,.035,.140,.175,.035,.140,.175)	.0032	<24	12121212121210	8	8	4						
(.024,.024,.032,.036,.036,.048,.096,.096,.128,.144,.144,.192)	.0031	25	12121212121211	8	6	5						
(.032,.032,.096,.048,.048,.144,.048,.048,.144,.072,.072,.216)	.0030	<24	12121212121111	9	8	2						
(.025,.075,.150,.025,.075,.150,.025,.075,.150,.025,.075,.150)	.0026	41	12121212	8	8	8	4	4	0			
(.024,.048,.048,.036,.072,.072,.056,.112,.112,.084,.168,.168)	.0021	<24	1212121212121010	7	2							
(.040,.060,.100,.040,.060,.100,.060,.090,.150,.060,.090,.150)	.0013	<24	1212121212121210	6	2							
(.045,.045,.060,.045,.045,.060,.105,.105,.140,.105,.105,.140)	.0013	<24	1212121212121210	6	0							
(.050,.100,.100,.050,.100,.100,.050,.100,.100,.050,.100,.100)	.0006	<24	12121212121212	4	0							
(.083,.083,.083,.083,.083,.083,.083,.083,.083,.083,.083,.083)	.0000	<24	12121212121212	0	0							
(.001,.001,.008,.009,.009,.072,.009,.009,.072,.081,.081,.648)	.0300	> 96	111111	9	7	7	7	7	7	7		
(.002,.004,.014,.008,.016,.056,.018,.036,.126,.072,.144,.504)	.0181	> 96	9	9	9	9	8	7	7	6	6	3
(.002,.002,.006,.018,.018,.054,.018,.018,.054,.162,.162,.486)	.0177	> 96	9	9	9	9	9	7	7	7	7	3
(.005,.005,.040,.005,.005,.040,.045,.360,.045,.045,.360)	.0156	> 96	101010101010	6	4	4	4					
(.004,.008,.028,.016,.032,.112,.016,.032,.112,.064,.128,.448)	.0139	> 96	8	8	8	8	7	7	7	5	4	2
(.003,.009,.018,.007,.021,.042,.027,.081,.162,.063,.189,.378)	.0113	> 96	9	9	9	8	7	7	6	6	4	3
(.004,.006,.010,.016,.024,.040,.036,.054,.090,.144,.216,.360)	.0107	> 96	9	9	8	8	8	7	7	5	4	3
(.004,.016,.020,.006,.024,.030,.036,.144,.180,.054,.216,.270)	.0080	> 96	8	8	8	8	7	7	6	4	2	
(.004,.016,.020,.006,.024,.030,.036,.144,.180,.054,.216,.270)	.0080	> 96	8	8	8	8	7	7	6	4	2	

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